Math 541 Lecture #35
V.1: Functions in $L^p(E)$ and their Norms
V.3: The Hölder and Minkowski Inequalities, Part I

V.1: Functions in $L^p(E)$ and their Norms. The development of the theory of L^p spaces is based in part on the notion of convexity.

Definition. A real-valued function φ defined on an open interval (a,b) is **convex** if

$$\varphi\big((1-\lambda)x+\lambda y\big) \le (1-\lambda)\varphi(x)+\lambda\varphi(y)$$

holds for all $x,y\in(a,b)$ and for all $\lambda\in[0,1]$.

Proposition. For a real $p\ge 1$, the function $\varphi(t)=t^p$ defined on $(0,\infty)$ is convex.

Homework Problem 35A. Give a proof of this Proposition.

The Space $L^p(E)$ for $p\in[1,\infty)$. Let $\{X,\mathcal{A},\mu\}$ be a measure space, $E\in\mathcal{A}$, and $p\ge 1$ a real number.

A measurable function $f:E\to\mathbb{R}^*$ is said to be in $L^p(E)$ if $|f|^p$ is integrable on E.

The $L^p(E)$ **norm** of a measurable function $f:E\to\mathbb{R}^*$ is

$$\|f\|_p = \left(\int_E |f|^p \, d\mu\right)^{1/p}.$$

A measurable function $f:E\to\mathbb{R}^*$ is in $L^p(E)$ if and only if $\|f\|_p<\infty$.

The L^p-norm satisfies $\|f\|_p\ge 0$, with $\|f\|_p=0$ if and only if $f=0$ a.e. in E.

The L^p-norm also satisfies for all $\alpha\in\mathbb{R}$,

$$\|\alpha f\|_p = \left(\int_E |\alpha f| \, d\mu\right)^{1/p} = |\alpha| \left(\int_E |f| d\mu\right)^{1/p} = |\alpha| \ \|f\|_p.$$

This says that the scalar multiple αf is in $L^p(E)$ for all $\alpha\in\mathbb{R}$ when $f\in L^p(E)$.

For $f,g\in L^p(E)$ and $\alpha,\beta\in\mathbb{R}$ we have by the convexity of $\varphi(t)=t^p$ on $(0,\infty)$ that

$$\left(\frac{|f|+|g|}{2}\right)^p \le \frac{|f|^p}{2}+\frac{|g|^p}{2}.$$

[When either $f(x)=0$ or $g(x)=0$, the inequality holds trivially.]

Moving the factor of $(1/2)^p$ from the left-hand side to the right-hand side gives

$$(|f|+|g|)^p \le 2^{p-1}\big(|f|^p+|g|^p\big).$$

By this and the triangle inequality we have that

$$\begin{aligned}\|f+g\|_p^p = \int_E |f+g|^p \, d\mu &\le \int_E \big(|f|+|g|\big)^p \, d\mu \\ &\le \int_E 2^{p-1}\left(|f|^p+|g|^p\right) \, d\mu = 2^{p-1}\|f\|_p^p + 2^{p-1}\|g\|_p^p.\end{aligned}$$

This shows that the sum of two $L^p(E)$ functions is an $L^p(E)$ function.

Proposition. For each $p \in [1, \infty)$, the set $L^p(E)$ is a linear space.

We will show later that $L^p(E)$ is a normed linear space (we haven't yet established the triangle inequality for the $L^p(E)$ norm).

The Space $L^\infty(E)$. A measurable function $f : E \to \mathbb{R}^*$ is said to be in $L^\infty(E)$ is there exists a positive real number M such that $|f(x)| \leq M$ for a.e. $x \in E$.

To define a "norm" on $L^\infty(E)$, we define for $f : E \to \mathbb{R}^*$ the quantity

$$\operatorname*{ess\,sup}_E f = \begin{cases} \inf\{k \in \mathbb{R} : \mu([f > k]) = 0\} & \text{if there is } k \in \mathbb{R} \text{ such that } \mu([f > k]) = 0, \\ \infty & \text{otherwise.} \end{cases}$$

This quantity is called the **essential supremum** of f.

The $L^\infty(E)$ **norm** of a measurable $f : E \to \mathbb{R}^*$ is

$$\|f\|_\infty = \operatorname*{ess\,sup}_E |f|.$$

A measurable function $f : E \to \mathbb{R}^*$ is in $L^\infty(E)$ if and only if $\|f\|_\infty < \infty$.

For $f \in L^\infty(E)$ the quantity $\|f\|_\infty$ is the unique real number such that for all $\epsilon > 0$ we have that

$$\mu(\{x \in E : |f(x)| \geq \|f\|_\infty + \epsilon\}) = 0,$$

and

$$\mu(\{x \in E : |f(x)| \geq \|f\|_\infty - \epsilon\}) > 0.$$

For $f \in L^\infty(E)$ and nonzero $\alpha \in \mathbb{R}$, we compute the value of $\|\alpha f\|_\infty$: for $\epsilon > 0$ we have

$$\begin{aligned} \{x \in E : |f(x)| \geq \|f\|_\infty + \epsilon/|\alpha|\} &= \{x \in E : |\alpha| \ |f(x)| \geq |\alpha| \ \|f\|_\infty + \epsilon\} \\ &= \{x \in E : |(\alpha f)(x)| \geq |\alpha| \ \|f\|_\infty + \epsilon\}, \end{aligned}$$

where the first and hence all the sets have measure zero, so that $\|\alpha f\|_\infty \leq |\alpha| \ \|f\|_\infty$; this shows that $\alpha f \in L^\infty(E)$; also we have

$$\begin{aligned} \{x \in E : |(\alpha f)(x)| \geq \|\alpha f\|_\infty + \epsilon\} &= \{x \in E : |\alpha| \ |f(x)| \geq \|\alpha f\|_\infty + \epsilon\} \\ &= \{x \in E : |f(x)| \geq \|\alpha f\|_\infty/|\alpha| + \epsilon/|\alpha|\}, \end{aligned}$$

where the first and hence all the sets have measure zero, so that $\|f\|_\infty \leq \|\alpha f\|_\infty/|\alpha|$.

Thus

$$\|\alpha f\|_\infty = |\alpha| \ \|f\|_\infty.$$

For $\alpha = 0$ we have that $|\alpha f(x)| = |\alpha| \ |f(x)| = 0 \cdot |f(x)| = 0$ for all $x \in E$, and so $\|\alpha f\|_\infty = 0 = |\alpha| \ \|f\|_\infty$.

Thus $\|\alpha f\|_\infty = |\alpha| \ \|f\|_\infty$ for all $\alpha \in \mathbb{R}$ and all $f \in L^\infty(E)$.

For $f \in L^\infty(E)$, the quantity $\|f\|_\infty$ is the smallest real number such that for all $\lambda \geq \|f\|_\infty$ we have

$$|f(x)| \leq \lambda \text{ for a.e. } x \in E.$$

The $L^\infty(E)$ norm satisfies $\|f\|_\infty \geq 0$, with $\|f\|_\infty = 0$ if and only if $f = 0$ a.e. in E.

For $f, g \in L^\infty(E)$ we have that

$$|f(x) + g(x)| \leq |f(x)| + |g(x)| \leq \|f\|_\infty + \|g\|_\infty < \infty,$$

for a.e. $x \in E$, so that $f + g \in L^\infty(E)$.

Proposition. The set $L^\infty(E)$ is a linear space.

We will show later that $L^\infty(E)$ is a normed linear space (we haven't yet established the triangle inequality for the $L^\infty(E)$ norm).

§3: The Hölder and Minkowski Inequalities. We show that the $L^p(E)$ norm $\|f\|_p$ satisfies the triangle inequality for all $1 \leq p \leq \infty$.

The case of $p = 1$ follows because

$$\begin{aligned}
\|f + g\|_1 &= \left(\int_E |f + g|^1 \, d\mu \right)^{1/1} \\
&= \int_E |f + g| \, d\mu \\
&\leq \int_E (|f| + |g|) \, d\mu \\
&= \int_E |f| \, d\mu + \int_E |g| \, d\mu \\
&= \left(\int_E |f|^1 \, d\mu \right)^{1/1} + \left(\int_E |g|^1 \, d\mu \right)^{1/1} \\
&= \|f\|_1 + \|g\|_1.
\end{aligned}$$

The case of $p = \infty$ follows because $\|f + g\|_\infty$ is the smallest real number such that

$$|f(x) + g(x)| \leq \|f + g\|_\infty$$

for a.e. $x \in E$, and

$$|f(x) + g(x)| \leq |f(x)| + |g(x)| \leq \|f\|_\infty + \|g\|_\infty$$

for a.e. $x \in E$, implying that

$$\|f + g\|_\infty \leq \|f\|_\infty + \|g\|_\infty.$$

Obtaining the cases $1 < p < \infty$ requires much more work.