Math 541 Lecture #36
V.3: The Hölder and Minkowski Inequalities, Part II

We continue towards a proof that the L^p "norms" satisfy the triangle inequality.

Two elements $p, q \in [1, \infty]$ are said to be **conjugate** if

$$\frac{1}{p} + \frac{1}{q} = 1,$$

where if $p = 1$ then $q = \infty$ and $1/q = 0$ because as $p \to 1$, we have $q \to \infty$, and if $q = 1$, then $p = \infty$ and $1/p = 0$ because as $q \to 1$ we have $p \to \infty$.

For example, $p = 2$ and $q = 2$ are conjugate.

Proposition 2.1. If $p, q \in [1, \infty]$ are conjugate, then for all $a, b \in \mathbb{R}$ we have

$$|ab| \leq \frac{|a|^p}{p} + \frac{|b|^q}{q}.$$

Proof. The inequality holds if $|a| = 0$ or $|b| = 0$ because $|ab| = 0$.

So suppose that $|a| > 0$ and $|b| > 0$.

The inequality holds if $p = 1$ because as $p \to 1$ we have $q \to \infty$ so that when $|b| \leq 1$ we have $|b|^q/q \to 0$ as $q \to \infty$, so that

$$|ab| = |a|\ |b| \leq |a| \leq \frac{|a|^1}{1} + \frac{|b|^\infty}{\infty},$$

and when $|b| > 1$ we have $|b|^q/q \to \infty$ as $q \to \infty$ (by L'Hospital's Rule), so that

$$|ab| < \infty = \frac{|a|^1}{1} + \frac{|b|^\infty}{\infty};$$

we have a similar conclusion when $q \to 1$.

So we suppose that $1 < p, q < \infty$.

The function

$$s \to \left(\frac{s^p}{p} + \frac{1}{q} - s \right), s \geq 0,$$

has an absolute minimum at $s = 1$ because its derivative

$$\frac{ps^{p-1}}{p} - 1 = s^{p-1} - 1$$

has a zero at $s = 1$, and its second derivative

$$(p-1)s^{p-2}$$

is positive on $s \geq 0$.

Hence for all $s \geq 0$ we have

$$\frac{1^p}{p} + \frac{1}{q} - 1 \leq \frac{s^p}{p} + \frac{1}{q} - s,$$

with equality holding only when $s = 1$.

Since p and q are conjugate we have

$$\frac{1^p}{p} + \frac{1}{q} - 1 = 0$$

so that

$$0 \leq \frac{s^p}{p} + \frac{1}{q} - s.$$

This rearranges to give

$$s \leq \frac{s^p}{p} + \frac{1}{q}$$

with equality holding only when $s = 1$.

Choosing

$$s = \frac{|a|}{|b|^{q/p}}$$

in the inequality gives

$$\frac{|a|}{|b|^{q/p}} \leq \frac{\left(\frac{|a|}{|b|^{q/p}}\right)^p}{p} + \frac{1}{q} = \frac{|a|^p}{p|b|^q} + \frac{1}{q}.$$

Multiplying the inequality through by $|b|^q$ gives

$$\frac{|a|\ |b|^q}{|b|^{q/p}} \leq \frac{|a|^p}{p} + \frac{|b|^q}{q}.$$

Here

$$\frac{|b|^q}{|b|^{q/p}} = |b|^{q-q/p}$$

where, because p and q are conjugate we have

$$\frac{1}{p} + \frac{1}{q} = 1 \ \Rightarrow\ \frac{q}{p} + 1 = q \ \Rightarrow\ q - \frac{q}{p} = 1,$$

we have $|b|^{q-q/p} = |b|$.

Therefore we obtain the inequality. $\square$

Proposition 2.2 (Hölder's Inequality). If $f \in L^p(E)$ and $g \in L^q(E)$ for conjugate p and q, then $fg \in L^1(E)$ and

$$\int_E |fg|\ d\mu \leq \|f\|_p \|g\|_q.$$

Moreover, equality holds only if there is a constant c such that $|f(x)|^p = c|g(x)|^q$ for a.e. $x \in E$.

Proof. If either $f = 0$ a.e. in E or $g = 0$ a.e. in E, there is nothing to show.

We assume WLOG that $f \geq 0$ and $g \geq 0$ with neither equal to 0 a.e. in E, so that $\|f\|_p \neq 0$ and $\|g\|_q \neq 0$.

For $p = 1$ and $q = \infty$ (similarly for $p = \infty$ and $q = 1$) we have

$$|fg| = |f|\ |g| \leq |f|\ \|g\|_\infty,$$

so that

$$\int_E |fg|\ d\mu \leq \int_E |f|\ \|g\|_\infty\ d\mu = \|g\|_\infty \int_E |f|\ d\mu = \|g\|_\infty \|f\|_1 = \|f\|_1 \|g\|_\infty.$$

For $p, q \in (1, \infty)$, if we set

$$a = \frac{f}{\|f\|_p},\ \ b = \frac{g}{\|g\|_q}$$

and substitute these into the inequality

$$|ab| \leq \frac{|a|^p}{p} + \frac{|b|^q}{q},$$

we get

$$\frac{fg}{\|f\|_p \|g\|_q} \leq \frac{f^p}{p\|f\|_p^p} + \frac{g^q}{q\|g\|_q^q} \text{ a.e. in } E.$$

Integrating over E gives

$$\begin{aligned}
\frac{1}{\|f\|_p \|g\|_q} \int_E fg\ d\mu &\leq \frac{1}{p\|f\|_p^p} \int_E f^p\ d\mu + \frac{1}{q\|g\|_q^q} \int_E g^q\ d\mu \\
&= \frac{\|f\|_p^p}{p\|f\|_p^p} + \frac{\|g\|_q^q}{q\|g\|_q^q} \\
&= \frac{1}{p} + \frac{1}{q} = 1.
\end{aligned}$$

Multiplication by $\|f\|_p \|g\|_q$ gives the inequality

$$\int_E fg\ d\mu \leq \|f\|_p \|g\|_q,$$

from which it follows that $fg \in L^1(E)$ when $f \in L^p(E)$ and $g \in L^p(E)$.

This inequality is derived from the inequality

$$s \leq \frac{s^p}{p} + \frac{1}{q}$$

for which equality holds only when $s = 1$.

Hence

$$1 = s = \frac{|a|}{|b|^{q/p}} \Rightarrow |a| = |b|^{q/p},$$

and since $a = f/\|f\|_p$ and $b = g/\|g\|_q$ we obtain

$$\frac{|f|}{\|f\|_p} = \frac{|g|^{q/p}}{\|g\|_q^{q/p}}.$$

Applying the p^{th} power to both sides gives

$$\frac{|f|^p}{\|f\|_p^p} = \frac{|g|^q}{\|g\|_q^q},$$

and hence that

$$|f|^p = \frac{\|f\|_p^p}{\|g\|_q^q}|g|^q.$$

Therefore equality holds only when $|f|^p = c|g|^q$ for $c = \|f\|_p^p/\|g\|_q^q$. $\square$