Math 541 Lecture #37
V.7: Convergence in $L^p(E)$ and Completeness, Part I

§7: Convergence in $L^p(E)$ and Completeness. We say that a sequence $\{f_n\}$ in $L^p(E)$ converges to $f \in L^p(E)$, written $\{f_n\} \to f$, if

$$\lim_{n\to\infty} \|f_n - f\|_p = 0.$$

This is convergence in the norm topology of $L^p(E)$.

A sequence $\{f_n\}$ in $L^p(E)$ is Cauchy if for every $\epsilon > 0$ there exists $N_\epsilon \in \mathbb{N}$ such that

$$\|f_n - f_m\|_p \leq \epsilon \text{ for all } n, m \geq N_\epsilon.$$

Proposition. If $\{f_n\} \to f$ in $L^p(E)$, then $\{f_n\}$ is Cauchy.

Proof. Suppose $\{f_n\} \to f$ in $L^p(E)$.

Then for $\epsilon > 0$ there exists $N_\epsilon \in \mathbb{N}$ such that $\|f_n - f\|_p \leq \epsilon/2$ for all $n \geq N_\epsilon$.

Hence by Minkowski's inequality we have for all $n, m \geq N_\epsilon$ that

$$\begin{aligned} \|f_n - f_m\|_p &= \|(f_n - f) - (f_m - f)\|_p \\ &\leq \|f_n - f\|_p + \|f_m - f\|_p \\ &\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

Thus $\{f_n\}$ is Cauchy in $L^p(E)$. □

Theorem 7.1 (Riesz-Fischer). For $1 \leq p \leq \infty$, if $\{f_n\}$ is a Cauchy sequence in $L^p(E)$, then there exists $f \in L^p(E)$ such that $\{f_n\} \to f$.

Proof. We first assume $1 \leq p < \infty$.

Suppose $\{f_n\}$ is Cauchy.

Then for each $j \in \mathbb{N}$, there is $n_j \in \mathbb{N}$ such that

$$\|f_n - f_m\|_p \leq \frac{1}{2^j} \text{ for all } n, m \geq n_j.$$

WLOG we assume that $n_j < n_{j+1}$ for all j.

Formally set

$$f(x) = f_{n_1}(x) + \sum_{j=1}^{\infty} \big(f_{n_{j+1}}(x) - f_{n_j}(x)\big) \text{ a.e. in } E.$$

We claim that this defines a function $f \in L^p(E)$ for which $\{f_n\} \to f$.

For $m \in \mathbb{N}$ set

$$g_m(x) = \sum_{j=1}^{m} |f_{n_{j+1}}(x) - f_{n_j}(x)| \text{ a.e. in } E.$$

Each g_m is nonnegative, and $\{g_m\}$ is monotone nondecreasing, i.e., $g_m(x) \leq g_{m+1}(x)$.

The limit function

$$g(x) = \lim_{m\to\infty} g_m(x) = \sup g_m(x)$$

exists a.e. in E, and is measurable.

Thus we have that

$$g^p(x) = \lim_{m\to\infty} g_m^p(x) = \liminf_{m\to\infty} g_m^p(x).$$

By Fatou's Lemma we have

$$\|g\|_p^p = \int_E g^p \, d\mu = \int_E \liminf g_m^p \, d\mu \le \liminf \int_E g_m^p \, d\mu = \liminf \|g_m\|_p^p.$$

By Minkowski's inequality and $n_{j+1} > n_j$ we have

$$\begin{aligned}
\|g_m\|_p &= \left\| \sum_{j=1}^{m} |f_{n_{j+1}} - f_{n_j}| \right\|_p \\
&\le \sum_{j=1}^{m} \|f_{n_{j+1}} - f_{n_j}\|_p \\
&\le \sum_{j=1}^{m} \frac{1}{2^j} \le 1.
\end{aligned}$$

Thus $\liminf \|g_m\|_p < \infty$, so that $\|g\|_p < \infty$, hence $g \in L^p(E)$ implying that $g(x) < \infty$ a.e. in E.

For those $x \in E$ where $g(x) < \infty$ the series

$$\sum_{j=1}^{\infty} \big(f_{n_{j+1}}(x) - f_{n_j}(x)\big)$$

converges absolutely because

$$g_m(x) = \sum_{j=1}^{m} |f_{n_{j+1}}(x) - f_{n_j}(x)|$$

converges to $g(x) < \infty$.

Since

$$f_{n_1} + \sum_{j=1}^{k-1} (f_{n_{j+1}} - f_{n_j}) = f_{n_k},$$

we obtain

$$f(x) = \lim_{j\to\infty} f_{n_j}(x) \text{ a.e. } x \in E.$$

Thus f is a measurable function.

Recall for an absolutely convergent series $\sum a_n$ that $|\sum a_n| \le \sum |a_n|$.

By this, we have $f \in L^p(E)$ because

$$|f(x)| \leq |f_{n_1}(x)| + g(x) \text{ a.e. in } E$$

and f_{n_1} and g are in $L^p(E)$.

It remains to show that pointwise limit f of $\{f_{n_j}\}$ is the L^p limit of $\{f_n\}$.

Recall that for each $j \in \mathbb{N}$ there is $n_j \in \mathbb{N}$ such that $\|f_n - f_m\|_p < 1/2^j$ for all $n, m \geq n_j$.

Thus for every $m \geq n_j$, we have by Fatou's Lemma

$$\begin{aligned}
\int_E |f - f_m|^p \, d\mu &= \int_E \lim_{k\to\infty} |f_{n_k} - f_m|^p \, d\mu \\
&= \int_E \liminf_k |f_{n_k} - f_m|^p \, d\mu \\
&\leq \liminf_k \int_E |f_{n_k} - f_m|^p \, d\mu \\
&\leq \liminf_k \|f_{n_k} - f_m\|_p^p \\
&\leq \|f_{n_j} - f_m\|_p^p \\
&\leq \left(\frac{1}{2^j}\right)^p.
\end{aligned}$$

This says that

$$\|f - f_m\|_p = \left(\int_E |f - f_m|^p \, d\mu\right)^{1/p} \leq \frac{1}{2^j}$$

which implies that

$$\|f - f_m\|_p \to 0 \text{ as } m \to \infty.$$

Therefore the Cauchy sequence $\{f_n\}$ in $L^p(E)$ converges to $f \in L^p(E)$.

The proof of the Riesz-Fischer Theorem when $p = \infty$ is in the next lecture.