Math 541 Lecture #38
V.7: Convergence in $L^p(E)$ and Completeness, Part II
V.8: Separating $L^p(E)$ by Simple Functions
V.18: If $E \subset \mathbb{R}^N$ and $p \in [1,\infty)$, then $L^p(E)$ is Separable, Part I

It remains to prove the Riesz-Fisher Theorem when $p = \infty$.

Let
$$A_k = \{x \in E : |f_k(x)| > \|f_k\|_\infty\}$$
and
$$B_{n,m} = \{x \in E : |f_n(x) - f_m(x)| > \|f_n - f_m\|_\infty\}.$$
Set
$$F = \left(\bigcup_{k=1}^\infty A_k\right) \cup \left(\bigcup_{n,m}^\infty B_{m,n}\right).$$

Because $f_k \in L^\infty(E)$ for all k we have $\mu(A_k) = 0$.

Because $f_n - f_m \in L^\infty(E)$ for all n, m, we have $\mu(B_{n,m}) = 0$.

Hence $\mu(F) = 0$.

For all $x \in E - F$ we have
$$|f_n(x) - f_m(x)| \le \|f_n - f_m\|_\infty \to 0.$$

This means that $\{f_n\}$ on $E - F$ is uniformly Cauchy, i.e., for $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $|f_n(x) - f_m(x)| < \epsilon$ for all $n, m > N$ and all $x \in E - F$.

Thus for each $x \in E - F$, the sequence $\{f_n(x)\}$ is Cauchy in $\mathbb{R}$.

By the completeness of $\mathbb{R}$, the sequence $\{f_n(x)\}$ converges to say $f(x)$.

For $x \in F$ define $f(x) = 0$.

We claim the function f is in $L^\infty(E)$ because it is bounded.

If
$$\sup\{\|f_m\|_\infty : m \in \mathbb{N}\} = \infty,$$
then we get a contradiction to
$$|\ \|f_n\|_\infty - \|f_m\|_\infty\ | \le \|f_n - f_m\|_\infty < \epsilon$$
for fixed $n \ge N$ and arbitrarily large $m > N$.

Thus
$$\sup\{\|f_m\|_\infty : m \in \mathbb{N}\} < \infty.$$

This means there is a finite $K > 0$ such that $|f_m(x)| \le K$ for all $x \in E - F$ and all m.

As $f_n(x) \to f(x)$ pointwise for $x \in E - F$ and $f(x) = 0$ for $x \in F$, we obtain
$$\sup\{|f(x)| : x \in E\} < \infty.$$

Thus $f \in L^\infty(E)$.

The sequence $\{f_n\}$ converges uniformly to f on $E - F$ because for $m > N$ we have

$$|f(x) - f_m(x)| = \lim_{n\to\infty} |f_n(x) - f_m(x)| \leq \|f_n - f_m\|_\infty < \epsilon,$$

so that $|f(x) - f_n(x)| < \epsilon$ for all $n > N$ uniformly in $x \in E - F$.

Thus $\|f - f_n\|_\infty < \epsilon$ for all $n > N$, i.e., f_n converges to f in $L^\infty(E)$. $\square$

Being complete as metric spaces, the $L^p(E)$ spaces are of second category, i.e., they are not the countable union of nowhere dense sets.

§8: Separating $L^p(E)$ by Simple Functions. We show that the simple functions in $L^p(E)$ are dense.

This does not say (yet) that $L^p(E)$ is separable (has a countable dense subset).

Proposition 8.1. If $f \in L^p(E)$ for some $1 \leq p \leq \infty$, then for every $\epsilon > 0$ there exists a simple function $\varphi \in L^p(E)$ such that $\|f - \varphi\|_p \leq \epsilon$.

Proof. By the decomposition $f = f^+ - f^-$ if we can find simple functions φ and ψ such that

$$\|f^+ - \varphi\|_p < \epsilon/2$$

and

$$\|f^- - \psi\|_p < \epsilon/2,$$

then $\varphi - \psi$ is a simple function for which

$$\begin{aligned} \|f - (\varphi - \psi)\|_p &= \|(f^+ - \varphi) + (\psi - f^-)\|_p \\ &\leq \|f^+ - \varphi\|_p + \|f^- - \psi\|_p \\ &\leq \epsilon. \end{aligned}$$

So we need only show that for a nonnegative f there is a simple function φ such that $\|f - \varphi\|_p \leq \epsilon$.

Since f is measurable, there is a sequence of simple functions $\{\varphi_n\}$ for which $\varphi_n \leq \varphi_{n+1}$ and $\varphi_n \to f$ every where in E.

For $1 \leq p < \infty$, the sequence $(f - \varphi_n)^p$ converges to zero a.e. in E, and $\{(f - \varphi_n)^p\}$ is dominated by the integrable f^p, i.e., $(f - \varphi_n)^p \leq f^p$.

By the Dominated Convergence Theorem we have

$$\lim_{n\to\infty} \|f - \varphi_n\|_p^p = \lim_{n\to\infty} \int_E (f - \varphi_n)^p \, d\mu = \int_E 0 \, d\mu = 0.$$

Thus for $\epsilon > 0$ there exists n such that $\|f - \varphi_n\|_p \leq \epsilon$.

For $p = \infty$, we have $\|f\|_\infty < \infty$ so that $f(x) \leq \|f\|_\infty$ a.e. in E.

By the construction of $\{\varphi_n\}$ we have $f(x) - \varphi_n(x) \leq 2^{-n}$ a.e. in E.

Thus

$$\mu\left(\{x \in E : f(x) - \varphi_n(x) > 2^{-n}\}\right) = 0.$$

This implies, since

$$\|f - \varphi_n\|_\infty = \inf\{k \in \mathbb{R} : \mu(\{x \in E : f(x) - \varphi_n(x) > k\} = 0\},$$

that

$$\|f - \varphi_n\|_\infty \leq 2^{-n},$$

and hence for $\epsilon > 0$ the choice of n such that $2^{-n} \leq \epsilon$ gives $\|f - \varphi_n\|_\infty \leq \epsilon$. □

§18: If $E \subset \mathbb{R}^N$ and $p \in [1, \infty)$, then $L^p(E)$ is Separable. We know that the collection of simple functions in $L^p(E)$ is dense in $L^p(E)$ for all $p \in [1, \infty]$.

The structure of $\mathbb{R}^N$ enables the separability of $L^p(E)$ when E is a Lebesgue measurable subset of $\mathbb{R}^N$ and $p \in [1, \infty)$.

When dealing with Lebesgue measure μ on $\mathbb{R}$ we typically replace the $d\mu$ in the Lebesgue integral with dx, i.e.,

$$\int_E f \ dx.$$

Theorem 18.1. Let $E \subset \mathbb{R}^N$ be Lebesgue measurable. The complete metric space $L^p(E)$ is separable when $p \in [1, \infty)$.

Proof. Let $\{Q_k\}$ be an enumeration of the closed dyadic cubes in $\mathbb{R}^N$.

For $n \in \mathbb{N}$, let S_n denote the collection of simple functions defined on E such that each $\varphi \in S_n$ has the form

$$\varphi = \sum_{i=1}^{n} a_i \chi_{Q_i \cap E}$$

for $a_i \in \mathbb{Q}$.

Each S_n is countable, and the union $S = \cup S_n$ is countable as well.

We show that S is dense in $L^p(E)$, i.e., for each $f \in L^p(E)$ there is for each $\epsilon > 0$ a $\varphi \in S$ such that $\|f - \varphi\|_p \leq \epsilon$.

To this end, we consider three cases: $\mu(E) = 0$, E is bounded with $\mu(E) > 0$ and $f \in L^\infty(E)$, and E is unbounded with $\mu(E) > 0$.

In the case of $\mu(E) = 0$, there is nothing to show because for every $\epsilon > 0$ we have for each $\varphi \in S_n$ that

$$\|f - \varphi\|_p^p = \int_E |f - \varphi|^p \ dx = 0 < \epsilon^p,$$

meaning that S_n is dense in $L^p(E)$.

The remaining two cases are in the next lecture.