Math 541 Lecture #39
V.18: If $E \subset \mathbb{R}^N$ and $p \in [1,\infty)$, then $L^p(E)$ is Separable, Part II

We continue with the proof of the Separability of $L^p(E)$ for $p \in [1,\infty)$.

Suppose E is bounded with $\mu(E) > 0$.

Let $f \in L^p(E) \cap L^\infty(E)$ with $\|f\|_\infty > 0$ (there is nothing to show if $\|f\|_\infty = 0$, because then $f = 0$ a.e. in E).

By Lusin's Theorem (need $f \in L^\infty(E)$ to apply this, i.e., f is a bounded function), the function f is quasi-continuous, so for $\epsilon > 0$ there is a closed set E_ϵ such that $E_\epsilon \subset E$,

$$\mu(E - E_\epsilon) \le \frac{\epsilon^p}{4^p \|f\|_\infty^p},$$

and f is continuous on E_ϵ.

Because E is bounded and E_ϵ is closed, the set E_ϵ is compact, so that f is uniformly continuous on E_ϵ.

Then for $\epsilon > 0$ there is $\delta > 0$ such that

$$|f(x) - f(y)| \le \frac{\epsilon}{4\mu(E)^{1/p}}$$

for all $x, y \in E_\epsilon$ with $|x - y| < \delta$.

Since E_ϵ is bounded, there are finitely many (depending on δ) closed dyadic cubes with pairwise disjoint interior and diameter less than δ whose union covers E_ϵ:

$$Q_1, Q_2, \dots, Q_{n_\delta},\ \mathring{Q}_i \cap \mathring{Q}_j = \emptyset \text{ for } i \ne j.$$

[Technically, we have a finite collection $Q_{k_1}, Q_{k_2}, \dots, Q_{k_{n_\delta}}$ of the dyadic cubes, not necessarily the first n_δ of them in the enumeration.]

Select $x_i \in Q_i \cap E_\epsilon$ for each $i = 1, 2, \dots, n_\delta$, and choose $a_i \in \mathbb{Q}$ such that $|a_i| \le |f(x_i)|$ and

$$|f(x_i) - a_i| \le \frac{\epsilon}{4\mu(E)^{1/p}}.$$

For the simple function

$$\varphi = \sum_{i=1}^{n_\delta} a_i \chi_{Q_i \cap E} \in S_n$$

we have

$$\int_E |f - \varphi|^p \, dx = \int_{E_\epsilon} |f - \varphi|^p \, dx + \int_{E - E_\epsilon} |f - \varphi|^p \, dx.$$

For the first integral on the right-hand side, because $E_\epsilon = \cup_{i=1}^{n_\delta} (Q_i \cap E_\epsilon)$ with pairwise intersections of measure zero, we have that

$$\begin{aligned}
\int_{E_\epsilon} |f-\varphi|^p \, dx &= \sum_{i=1}^{n_\delta} \int_{Q_i \cap E_\epsilon} |f-\varphi|^p \, dx = \sum_{i=1}^{n_\delta} \int_{Q_i \cap E_\epsilon} |f-a_i|^p \, dx \\
&= \sum_{i=1}^{n_\delta} \int_{Q_i \cap E_\epsilon} |(f-f(x_i)) + (f(x_i)-a_i)|^p \, dx \\
&\le \sum_{i=1}^{n_\delta} \int_{Q_i \cap E_\epsilon} 2^{p-1}\big(|f-f(x_i)|^p + |f(x_i)-a_i|^p\big) \, dx \\
&\le 2^{p-1} \sum_{i=1}^{n_\delta} \int_{Q_i \cap E_\epsilon} \left(\frac{\epsilon}{4\mu(E)^{1/p}}\right)^p \, dx + 2^{p-1} \sum_{i=1}^{n_\delta} \int_{Q_i \cap E_\epsilon} \left(\frac{\epsilon}{4\mu(E)^{1/p}}\right)^p \, dx \\
&= 2^p \sum_{i=1}^{n_\delta} \frac{\epsilon^p \mu(Q_i \cap E_\epsilon)}{4^p \mu(E)} \\
&= \frac{\epsilon^p \mu(E_\epsilon)}{2^p \mu(E)} \\
&\le \frac{\epsilon^p}{2^p}.
\end{aligned}$$

For the second integral on the right-hand side, because $\|\varphi\|_\infty = \max\{|a_i|\} \le \|f\|_\infty$ (this follows since $|a_i| \le |f(x_i)|$), we have that

$$\begin{aligned}
\int_{E-E_\epsilon} |f-\varphi|^p \, dx &\le \int_{E-E_\epsilon} 2^{p-1}\big(|f|^p + |\varphi|^p\big) \, dx \\
&\le \int_{E-E_\epsilon} 2^{p-1}\big(\|f\|_\infty^p + \|f\|_\infty^p\big) \, dx \\
&= 2^p \int_{E-E_\epsilon} \|f\|_\infty^p \, dx \\
&= 2^p \|f\|_\infty^p \mu(E-E_\epsilon) \\
&\le 2^p \|f\|_\infty^p \frac{\epsilon^p}{4^p \|f\|_\infty^p} \\
&= \frac{\epsilon^p}{2^p}.
\end{aligned}$$

Thus we have that

$$\int_E |f-\varphi|^p \, dx \le \frac{\epsilon^p}{2^p} + \frac{\epsilon^p}{2^p} = \frac{2\epsilon^p}{2^p} = \frac{\epsilon^p}{2^{p-1}} \le \epsilon^p$$

so that

$$\|f-\varphi\|_p = \left(\int_E |f-\varphi|^p \, dx\right)^{1/p} \le \epsilon.$$

Now assume that E is arbitrary with $\mu(E) > 0$.

For $f \in L^p(E)$ we have that

$$\sum_{n=1}^{\infty} \int_{E\cap\{n<|x|\leq n+1\}} |f|^p \ dx = \int_E |f|^p \ dx < \infty.$$

Thus for $\epsilon > 0$ there is $n_\epsilon \in \mathbb{N}$ such that

$$\int_{E\cap\{|x|\geq n_\epsilon\}} |f|^p \ dx \leq \frac{\epsilon^p}{4^p}$$

(the tail of the series goes to 0).

Also, since $f \in L^p(E)$, we also have for each $n \in \mathbb{N}$ that

$$\begin{aligned} n^p \mu([|f| \geq n]) &= \int_{[|f|\geq n]} n^p \ dx \\ &\leq \int_{[|f|\geq n]} |f|^p dx + \int_{[|f|<n]} |f|^p \ dx \\ &= \int_E |f|^p \ dx \\ &= \|f\|_p^p < \infty. \end{aligned}$$

Thus for every $\delta > 0$ there is $n_\delta \in \mathbb{N}$ such that

$$\mu([|f| \geq n]) \leq \delta \text{ for all } n \geq n_\delta$$

(the sequence $\{n^p\} \to \infty$ as $n \to \infty$, so that $\mu([|f|] \geq n]) \to 0$).

By Vitali's Theorem (the absolute continuity of the integral), for $\epsilon > 0$ there is $\delta > 0$ such that

$$\int_{[|f|\geq n]} |f|^p \ dx \leq \frac{\epsilon^p}{4^p} \text{ for all } n \geq n_\delta.$$

For the measurable set

$$E_n = (E \cap [|x| \geq n]) \cup [|f| \geq n],$$

and for $n \geq \max\{n_\epsilon, n_\delta\}$ we have that

$$\begin{aligned} \int_{E_n} |f|^p \ dx &\leq \int_{E\cap[|x|\geq n]} |f|^p \ dx + \int_{[|f|\geq n]} |f|^p \ dx \\ &\leq \frac{\epsilon^p}{4^p} + \frac{\epsilon^p}{4^p} = \frac{2\epsilon^p}{4^p}. \end{aligned}$$

Hence for all $n \geq \max\{n_\epsilon, n_\delta\}$, we have

$$\left(\int_{E_n} |f|^p \ dx \right)^{1/p} \leq \frac{2^{1/p}\epsilon}{4} \leq \frac{\epsilon}{2}.$$

Now the set $E - E_n$ is bounded because it is a subset of $[|x| < n]$, and the restriction of f to $E - E_n$ is bounded because $[|f| < n]$ on $E - E_n$.

By the previous case there is a simple function $\varphi \in S$ that is zero outside of $E - E_n$ (in particular it is zero on E_n) such that

$$\left(\int_{E-E_n} |f-\varphi|^p \, dx\right)^{1/p} dx \le \frac{\epsilon}{2}.$$

Therefore

$$\begin{aligned}
\|f-\varphi\|_p &= \|(f-\varphi)(\chi_{E_n} + \chi_{E-E_n})\|_p \\
&\le \|(f-\varphi)\chi_{E_n}\|_p + \|(f-\varphi)\chi_{E-E_n}\|_p \\
&= \left(\int_{E_n} |f-\varphi|^p \, dx\right)^{1/p} + \left(\int_{E-E_n} |f-\varphi|^p \, dx\right)^{1/p} \\
&= \left(\int_{E_n} |f-0|^p \, dx\right)^{1/p} + \left(\int_{E-E_n} |f-\varphi|^p \, dx\right)^{1/p} \\
&\le \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\end{aligned}$$

Thus the countable S is dense in $L^p(E)$, and hence $L^p(E)$ is separable. □

Recall that a topological space satisfies the second axiom of countability it the topology has a countable base.

Corollary. For each $p \in [1, \infty)$, the complete metric space $L^p(E)$ satisfies the second axiom of countability.

Proof. By Proposition 13.2 in Chapter I, a topological space satisfies the second axiom of countability if and only if it is separable.

By Theorem 18.1, the topological space $L^p(E)$ is separable. □