Math 541 Lecture #40
V.18: If $E \subset \mathbb{R}^N$ and $p \in [1,\infty)$, then $L^p(E)$ is Separable, Part III
Relationship Among $L^p(E)$ Spaces, Part I

§18.1: $L^\infty(E)$ is not Separable. Unfortunately, the topological space $L^\infty(E)$, for a Lebesgue measurable subset E of $\mathbb{R}^N$ of positive measure, is not separable; in this sense, it is too big of a space.

We will demonstrate this lack of separability when $E = \mathbb{R}^N$.

Proposition 18.3. The complete metric space $L^\infty(\mathbb{R}^N)$ is not separable.

Proof. For $s > 0$, let $B_s(0)$ be the open ball of radius s centered at the origin 0 in $\mathbb{R}^N$.

Then for $s > r > 0$ we have for every $0 < \epsilon \le 1$ that

$$\begin{aligned}\mu(\{x \in \mathbb{R}^N : |\chi_{B_s(0)}(x) - \chi_{B_r(0)}(x)| > 1-\epsilon\}) \\ &= \mu\big(B_s(0) - B_r(0)\big) \\ &= \mu(B_s(0)) - \mu(B_r(0)) > 0.\end{aligned}$$

This implies that

$$\begin{aligned}&\|\chi_{B_s(0)} - \chi_{B_r(0)}\|_\infty \\ &\quad = \inf\{k \in \mathbb{R} : \mu(\{x \in \mathbb{R}^N : |\chi_{B_s(0)}(x) - \chi_{B_r(0)}(x)| > k\}) = 0\} \\ &\quad \ge 1.\end{aligned}$$

[What is $\|\chi_{B_s(0)} - \chi_{B_r(0)}\|_p$ for $p \in [1,\infty)$? It is $\mu(B_s(0)) - \mu(B_r(0))$ which goes to zero as $s \to r$.]

Consider the uncountable collection

$$S = \{\chi_{B_s(0)} : s > 0\} \subset L^\infty(\mathbb{R}^N),$$

for which we have that for every $f, g \in S$ with $f \ne g$ there holds $\|f - g\|_\infty \ge 1$.

Suppose there is a countable subset M of $L^\infty(\mathbb{R}^N)$ for which $\overline{M} = L^\infty(\mathbb{R}^N)$.

Then for each $f \in L^\infty(\mathbb{R}^N)$ we have by the denseness of M that for every open ball

$$B_\epsilon(f) = \{g \in L^\infty(\mathbb{R}^N) : \|f - g\|_\infty < \epsilon\}$$

there exists $h \in M$ such that $h \in B_\epsilon(f)$.

In particular, this means that for each $f \in L^\infty(\mathbb{R}^N)$ there exists an $h \in M$ such that $f \in B_{1/3}(h) = \{f \in L^\infty(\mathbb{R}^N) : \|f - h\|_\infty < 1/3\}$.

Thus by the denseness of M in $L^\infty(\mathbb{R}^N)$ we have

$$L^\infty(\mathbb{R}^N) = \bigcup_{h \in M} B_{1/3}(h).$$

For each $h \in M$ the set $B_{1/3}(h) \cap S$ consists of at most one element of S because for distinct $f, g \in S$, if $\|f-h\|_\infty < 1/3$ and $\|g-h\|_\infty < 1/3$, we would have the contradiction

$$1 \leq \|f-g\|_\infty \leq \|f-h\|_\infty + \|h-g\|_\infty \leq 1/3 + 1/3 < 1.$$

Thus at most only countably many of the elements of S are in

$$\bigcup_{h \in M} B_{1/3}(h) = L^\infty(\mathbb{R}^N).$$

But S is uncountable, so uncountable many of the elements of S are not in this union, giving a contradiction to the assumed existence of a countably dense subset. □

Corollary. The complete metric space $L^\infty(\mathbb{R}^N)$ is does not satisfy the second axiom of countability, but does satisfies the first axiom of countability.

Proof. By Proposition 18.3 and Proposition 13.2 of Chapter I, the topological space $L^\infty(\mathbb{R}^N)$ does not satisfy the second axiom of countability.

However, at each element $f \in L^\infty(\mathbb{R}^N)$ there is a countable base at f, namely the balls $B_r(f)$ where $r \in \mathbb{Q}^+$. □

Relationships among the $L^P(E)$ Spaces. We will describe several relationships among the $L^p(E)$ spaces, starting with how the L^∞ norm fits in with the L^p norms.

Proposition. If $f \in L^r(E) \cap L^\infty(E)$ for some $r \in [1, \infty)$, and $\|f\|_\infty > 0$, then $f \in L^p(E)$ for all $p \in (r, \infty)$ where

$$\|f\|_p \leq \|f\|_\infty^{1-r/p} \|f\|_r^{r/p},$$

and

$$\lim_{p \to \infty} \|f\|_p = \|f\|_\infty.$$

Proof. For $\epsilon > 0$ set

$$V_\epsilon = \{x \in E : |f(x)| > \|f\|_\infty - \epsilon\}.$$

Since $\|f\|_\infty < \infty$, we have $\mu(V_\epsilon) > 0$.

For any $p \in [r, \infty)$ we have

$$\begin{aligned} \|f\|_p^p &= \int_E |f|^p \, d\mu \geq \int_{V_\epsilon} |f|^p \, d\mu \\ &\geq \int_{V_\epsilon} \big(\|f\|_\infty - \epsilon\big)^p d\mu = \big(\|f\|_\infty - \epsilon\big)^p \mu(V_\epsilon). \end{aligned}$$

For $p = r$ this gives

$$\infty > \|f\|_r^r \geq \big(\|f\|_\infty - \epsilon\big)^r \mu(V_\epsilon).$$

This implies that $\mu(V_\epsilon) < \infty$, and hence we have that $0 < \mu(V_\epsilon) < \infty$.

For arbitrary $r < p < \infty$, we have

$$\|f\|_p \geq \big(\|f\|_\infty - \epsilon\big) \big[\mu(V_\epsilon)\big]^{1/p}.$$

On the other hand,

$$\begin{aligned}\|f\|_p^p &= \int_E |f|^p \, d\mu = \int_E |f|^{p-r}|f|^r \, d\mu \\ &\leq \int_E \|f\|_\infty^{p-r}|f|^r \, d\mu = \|f\|_\infty^{p-r}\int_E |f|^r \, d\mu \\ &= \|f\|_\infty^{p-r}\|f\|_r^r.\end{aligned}$$

This implies that

$$\|f\|_p \leq \|f\|_\infty^{1-r/p}\|f\|_r^{r/p}.$$

Since

$$\lim_{p\to\infty}\left[\|f\|_\infty^{1-r/p}\|f\|_r^{r/p}\right] = \|f\|_\infty,$$

the set

$$\{\|f\|_p : r \leq p < \infty\}$$

is bounded, and so $f \in L^p(E)$ for all $p \in (r, \infty)$.

Since $\|f\|_p \geq \left(\|f\|_\infty - \epsilon\right)\left[\mu(V_\epsilon)\right]^{1/p}$ and $0 < \mu(V_\epsilon) < \infty$, we obtain

$$\liminf_{p\to\infty}\|f\|_p \geq \|f\|_\infty - \epsilon.$$

Since $\|f\|_p \leq \|f\|_\infty^{1-r/p}\|f\|_r^{r/p}$ and $\lim_{p\to\infty}\|f\|_\infty^{1-r/p}\|f\|_r^{r/p} = \|f\|_\infty$, we obtain

$$\limsup_{n\to\infty}\|f\|_p \leq \|f\|_\infty.$$

Thus for all $0 < \epsilon < \|f\|_\infty$, we have

$$\|f\|_\infty - \epsilon \leq \liminf_{p\to\infty}\|f\|_p \leq \limsup_{p\to\infty}\|f\|_p \leq \|f\|_\infty.$$

Therefore we obtain

$$\lim_{p\to\infty}\|f\|_p = \|f\|_\infty,$$

which completes the proof. $\square$

Corollary. If $f \in L^1(E)\cap L^\infty(E)$, then $f \in L^p(E)$ for all $p \in (0,\infty)$.