Math 541 Lecture #41
Relationship Among $L^p(E)$ Spaces, Part II
Final Exam Study Guide

Recall the containment $L^r(E) \cap L^\infty(E) \subset L^p(E)$ for all $p \in (r, \infty)$ where each $f \in L^r(E) \cap L^\infty(E)$ satisfies

$$\|f\|_p \leq \|f\|_r^{r/p} \|f\|_\infty^{1-r/p}.$$

We can generalize this containment.

Proposition. If $1 \leq r < p < q \leq \infty$, then $L^r(E) \cap L^q(E) \subset L^p(E)$ and each $f \in L^r(E) \cap L^q(E)$ satisfies

$$\|f\|_p \leq \|f\|_r^\lambda \|f\|_q^{1-\lambda},$$

where λ satisfies

$$\frac{1}{p} = \frac{\lambda}{r} + \frac{1-\lambda}{q}.$$

Proof. We have already proven the case of $q = \infty$, so we suppose that $q < \infty$.

Let $f \in L^r(E) \cap L^q(E)$, and define λ by

$$\frac{1}{p} = \frac{\lambda}{r} + \frac{1-\lambda}{q}.$$

Explicitly,

$$\lambda = \frac{\frac{1}{p} - \frac{1}{q}}{\frac{1}{r} - \frac{1}{q}} > 1,$$

where the inequality follows from $r < p < q$.

[Note that $q = \infty$ gives $\lambda = r/p$, matching $\|f\|_p \leq \|f\|_r^{r/p} \|f\|_\infty^{1-r/p}$.]

The quantities

$$\frac{r}{\lambda p} = \frac{1 - \frac{r}{q}}{1 - \frac{p}{q}} > 1 \text{ and } \frac{q}{(1-\lambda)p} = \frac{q}{p}\left(\frac{\frac{1}{r} - \frac{1}{q}}{\frac{1}{r} - \frac{1}{p}}\right) = \frac{\frac{q}{r} - 1}{\frac{p}{r} - 1} > 1$$

are conjugate because

$$\left(\frac{r}{\lambda p}\right)^{-1} + \left(\frac{q}{(1-\lambda)p}\right)^{-1} = \frac{\lambda p}{r} + \frac{(1-\lambda)p}{q} = p\left(\frac{\lambda}{r} + \frac{1-\lambda}{q}\right) = p\left(\frac{1}{p}\right) = 1.$$

The function $|f|^{\lambda p}$ is in $L^{r/\lambda p}(E)$ because

$$\int_E \left(|f|^{\lambda p}\right)^{r/\lambda p} d\mu = \int_E |f|^r < \infty,$$

and the function $|f|^{(1-\lambda)p}$ is in $L^{q/(1-\lambda)p}(E)$ because

$$\int_E \left(|f|^{(1-\lambda)p}\right)^{q/(1-\lambda)p} d\mu = \int_E |f|^q \, d\mu < \infty.$$

We apply Hölder's Inequality using the conjugate $r/\lambda p$ and $1/(1-\lambda)p$:

$$\begin{aligned}
\int_E |f|^p \, d\mu &= \int_E |f|^{\lambda p} |f|^{(1-\lambda)p} \, d\mu \\
&\leq \left(\int_E \left(|f|^{\lambda p}\right)^{r/\lambda p} d\mu \right)^{\lambda p/r} \left(\int_E \left(|f|^{(1-\lambda)p}\right)^{q/(1-\lambda)p} d\mu \right)^{(1-\lambda)p/q} \\
&= \left(\int_E |f|^r \, d\mu \right)^{\lambda p/r} \left(\int_E |f|^q \, d\mu \right)^{(1-\lambda)p/q} \\
&= \|f\|_r^{\lambda p} \|f\|_q^{(1-\lambda)p}.
\end{aligned}$$

Taking p^{th} roots gives

$$\|f\|_p \leq \|f\|_r^{\lambda} \|f\|_q^{1-\lambda},$$

which shows that $L^r(E) \cap L^q(E) \subset L^p(E)$. $\square$

Proposition. If $\mu(E) < \infty$ and $1 \leq p < q \leq \infty$, then $L^q(E) \subset L^p(E)$ and each $f \in L^q(E)$ satisfies

$$\|f\|_p \leq \|f\|_q [\mu(E)]^{(1/p)-(1/q)}.$$

Proof. Let $f \in L^q(E)$.

For $q = \infty$ we have

$$\|f\|_p^p = \int_E |f|^p \, d\mu \leq \int_E \|f\|_\infty^p \, d\mu = \|f\|_\infty^p \mu(E) < \infty,$$

so that $f \in L^p(E)$ and

$$\|f\|_p \leq \|f\|_\infty [\mu(E)]^{1/p}.$$

Now suppose $q < \infty$.

The quantities

$$\frac{q}{p} > 1 \text{ and } \frac{q}{q-p} > 1$$

are conjugate because

$$\left(\frac{q}{p}\right)^{-1} + \left(\frac{q}{q-p}\right)^{-1} = \frac{p}{q} + \frac{q-p}{q} = \frac{q}{q} = 1.$$

The function $|f|^p$ belongs to $L^{q/p}(E)$ because

$$\int_E \left(|f|^p\right)^{q/p} d\mu = \int_E |f|^q \, d\mu < \infty.$$

The function χ_E belongs to $L^{q/(q-p)}(E)$ because

$$\int_E |\chi_E|^{q/(q-p)} \, d\mu = \mu(E) < \infty.$$

We apply Hölder's Inequality with the conjugate q/p and $q/(q-p)$:

$$\begin{aligned}
\|f\|_p^p &= \int_E |f|^p \, d\mu \\
&= \int_E |f|^p \cdot \chi_E \, d\mu \\
&\leq \left(\int_E (|f|^p)^{q/p} d\mu\right)^{p/q} \left(\int_E |\chi_E|^{q/(q-p)} d\mu\right)^{(q-p)/q} \\
&= \|f\|_q^p [\mu(E)]^{(q-p)/q}.
\end{aligned}$$

Taking p^{th} roots gives

$$\|f\|_p \leq \|f\|_q [\mu(E)]^{(1/p)-(1/q)},$$

which shows that $L^q(E) \subset L^p(E)$. □

Final Exam Study Guide. The Final Exam, held on Thursday December 21 from 11 a.m. to 2 p.m. in 108 TMCB (in-class final), will have three parts.

Part I. Give statements and proofs of two of the following five named theorems.

(1) Fatou's Lemma (you may assume the Monotone Convergence Theorem in your proof).
(2) Lebesgue's Dominated Convergence Theorem (you may assume Fatou's Lemma in your proof).
(3) Vitali's Absolute Continuity Theorem (you may assume the Monotone Convergence Theorem in your proof).
(4) Hölder's Inequality (you may assume Young's Inequality in your proof).
(5) The Riesz-Fischer Theorem for $L^\infty(E)$.

Part II. Answer 4 of 6 homework-like problems (in fact, some of these may indeed be homework problems).

Part III. Answer two challenge problems, each with a hint.