Differential Operators and Entire Functions with Simple Real Zeros

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Abstract

Let ϕ and f be functions in the Laguerre-Pólya class. Write $\phi(z) = e^{-\alpha z^2}\phi_1(z)$ and $f(z) = e^{-\beta z^2} f_1(z)$, where ϕ_1 and f_1 have genus 0 or 1 and $\alpha, \beta \geq 0$. If $\alpha\beta < 1/4$ and ϕ has infinitely many zeros, then $\phi(D)f(z)$ has only simple real zeros, where D denotes differentiation.

Key words: differential operators, zeros of entire functions, Laguerre-Pólya class, simple zeros

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1 Introduction

In this paper we answer a question of Craven and Csordas stated in [1] regarding the simplicity of the zeros of $\phi(D)f(z)$, where both ϕ and f are in the Laguerre-Pólya class and D denotes differentiation. The Laguerre-Pólya class, denoted $\mathcal{LP}$, consists of the entire functions having only real zeros with Weierstrass products of the form

$$cz^m e^{\alpha z - \beta z^2} \prod_k \left(1 - \frac{z}{\alpha_k}\right) e^{z/\alpha_k},$$

where c, α, β, α_k are real, $\beta \geq 0$, $\alpha_k \neq 0$, m is a nonnegative integer, and $\sum_{k=1}^{\infty} 1/\alpha_k^2 < \infty$. An entire function belongs to $\mathcal{LP}$ if and only if it is the uniform limit on compact sets of a sequence of real polynomials having only real zeros [2, Thm. 3, p. 331].

One of the reasons for studying the Laguerre-Pólya class is its relationship to

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the Riemann zeta function. Let

$$\xi(s) = \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s),$$

where $\zeta(s)$ is the Riemann zeta function. Then $\xi(1/2+iz)$ is an even entire function of genus 1 that is real for real z. The Riemann hypothesis, which predicts that the zeros of $\xi(s)$ have real part 1/2, can be stated as $\xi(1/2+iz) \in \mathcal{LP}$. Furthermore, evidence suggests that most, if not all, of the zeros of $\xi(s)$ are simple. Hence, functions in $\mathcal{LP}$ with simple zeros are especially interesting.

For $\phi(z) = \sum_{k=0}^{\infty} a_k z^k \in \mathcal{LP}$ and $f \in \mathcal{LP}$ we consider the differential operator $\phi(D)$ defined by

$$\phi(D)f(z) = \sum_{k=0}^{\infty} a_k f^{(k)}(z).$$

With suitable hypotheses $\phi(D)f(z) \in \mathcal{LP}$ (see Lemma 2 below). There are several cases in which the zeros of $\phi(D)f(z)$ are known to be simple. Craven and Csordas proved that if ϕ and f have orders less than 2, if ϕ has infinitely many zeros, and if there is a bound on the multiplicity of the zeros of f, then $\phi(D)f(z)$ has only simple real zeros [1, Thm. 4.6]. They also showed that if ϕ and f have orders less than 2, if ϕ has infinitely many zeros, and if the canonical product representation of ϕ has genus zero, then $\phi(D)f(z)$ has only simple real zeros [1, Thm. 4.7]. In the same paper they state the open problem of whether $\phi(D)f(z)$ has simple zeros without the extra hypothesis bounding the order of zeros of f or the hypothesis that ϕ has genus zero [1, p. 819].

In this paper we answer that question in the affirmative with the following theorem:

Theorem 1 *Let ϕ and f be in $\mathcal{LP}$. Write $\phi(z) = e^{-\alpha z^2}\phi_1(z)$ and $f(z) = e^{-\beta z^2} f_1(z)$, where ϕ_1 and f_1 have genus* 0 *or* 1 *and $\alpha, \beta \geq 0$. If $\alpha\beta < 1/4$ and ϕ has infinitely many zeros, then $\phi(D)f(z)$ has only simple real zeros.*

This theorem is proved in §3.

We remark that the hypothesis $\alpha\beta < 1/4$ in Theorem 1 is necessary. The term 1/4 arises in proving the convergence of the series defining $\phi(D)f(z)$ as in Lemma 3.1 in [1, p. 806] or Theorem 8 in [2, p. 360]. On the other hand, if the Weierstrass product for ϕ contains the genus two factor $e^{-\alpha z^2}$ and if f has order less than two, the assumption that ϕ has infinitely many zeros is not necessary. Theorem 3.10 in [1] states that if $\alpha > 0$ and if g is a function in $\mathcal{LP}$ of order less than 2, then the zeros of $e^{-\alpha D^2} g(z)$ are simple and real. Consequently, if $\phi(z) = e^{-\alpha z^2}\phi_1(z)$ where $\alpha > 0$ and $\phi_1(z)$ has genus less than two, then $\phi(D)f(z) = e^{-\alpha D^2}\big(\phi_1(D)f(z)\big)$ has only simple zeros even if $\phi_1(z)$ has finitely many zeros. If ϕ lacks the genus two factor $e^{-\alpha z^2}$ and has finitely many zeros, the conclusion of the theorem does not hold.

2 Preliminaries

For $\phi(z) = \sum_{k=0}^{\infty} a_k z^k \in \mathcal{LP}$ and $f \in \mathcal{LP}$ it is important to know when the expression

$$\phi(D)f(z) = \sum_{k=0}^{\infty} a_k f^{(k)}(z)$$

makes sense. For our purposes, the following well known result will suffice.

Lemma 2 *Write $\phi(z) = e^{-\alpha z^2}\phi_1(z)$ and $f(z) = e^{-\beta z^2} f_1(z)$, where $\phi_1(z)$ and $f_1(z)$ have genus 0 or 1 and $\alpha, \beta \geq 0$. If $\alpha\beta < 1/4$, then $\phi(D)f(z) \in \mathcal{LP}$.*

Proof. See Levin [2, Thm. 8, p. 360]. □

Lemma 2 shows that under the assumptions of Theorem 1 the expression $\phi(D)f(z)$ represents a function in the Laguerre-Pólya class. Thus, $\phi(D)f(z)$ has only real zeros. A natural question to ask is whether the zeros are also simple. As the convergence of the sum defining $\phi(D)f(z)$ is not in question, the proof of Theorem 1 in the following section focuses solely on the question of simplicity.

3 Proof of Theorem 1

In this section we will prove Theorem 1. The proof builds upon results from the paper of Craven and Csordas [1] and upon well known facts about entire function as in Levin [2].

The basic outline of the proof of Theorem 1 is as follows: We begin by studying the effect of individual factors in the Weierstrass product for $\phi(D)$ on $f(z)$. Thus, in Lemmas 3 through 5, we consider the expression $h = f - \alpha^{-1}f'$. We show that if h has a zero of order $m \geq 2$ at x_0, then f has a zero of order at least $m+1$ at x_0. This fact will be used to prove Lemma 6 which says that in a fixed interval the expression $\prod_{k=1}^{n}(1 - \frac{D}{\alpha_k})f(z)$ has only simple zeros for sufficiently large n. This result is extended in Lemma 7 through Lemma 10 to show that if $\phi(z) = \prod_{k=1}^{\infty}(1 - \frac{z}{\alpha_k})$ is of genus 0, then $\phi(D)f(z)$ has only simple real zeros. Finally, in Lemma 11 the result is extended to the more general case, stated in the hypotheses of Theorem 1, to show that $\phi(D)f(z)$ has only simple real zeros. This proves Theorem 1. We will now proceed with the proof.

Lemma 3 *Let $f \in \mathcal{LP}$ and let $\alpha \neq 0$ be real. Then*

(1) $f' \in \mathcal{LP}$, and

(2) $h = (I - \alpha^{-1}D)f = f - \alpha^{-1}f' \in \mathcal{LP}$.

Proof. Although this is a special case of Lemma 2, we recall the elementary argument. Since f is the uniform limit of a sequence of real polynomials $\{f_n\}$ having only real zeros, f' is the uniform limit of the sequence $\{f_n'\}$. Because each f_n has only real zeros, each f_n' also has only real zeros. Hence, the zeros of f' are also real, and $f' \in \mathcal{LP}$. Then

$$h(z) = -\alpha^{-1}e^{\alpha z}D\Big(e^{-\alpha z}f(z)\Big).$$

So, h is also in $\mathcal{LP}$. $\square$

Lemma 4 (Laguerre Inequalities) *Let $f \in \mathcal{LP}$. Then*

$$\Big(f^{(n)}(z)\Big)^2 - f^{(n-1)}(z)f^{(n+1)}(z) \geq 0, \qquad -\infty < z < \infty, \quad n \geq 1.$$

Equality holds if and only if $f^{(n-1)}(z)$ is of the form $ce^{\alpha z}$ or if z is a multiple root of $f^{(n-1)}(z)$.

Proof. We follow the explanation in [3, p. 69]. If $f(z)$ is of the form $f(z) = ce^{\alpha z}$, then $[f'(z)]^2 - f(z)f''(z) = 0$ for all z. Otherwise, we express $f(z)$ as a Weierstrass product:

$$f(z) = cz^m e^{\alpha z - \beta z^2} \prod_k (1 - z/\alpha_k)e^{z/\alpha_k}.$$

The logarithmic derivative of $f(z)$ is

$$\frac{f'(z)}{f(z)} = \frac{m}{z} + \alpha - 2\beta z + \sum_{k=0}^{\infty}\left(\frac{1}{z-\alpha_k} + \frac{1}{\alpha_k}\right).$$

Hence,

$$\frac{d}{dz}\left(\frac{f'(z)}{f(z)}\right) = \frac{f''(z)f(z) - \Big(f'(z)\Big)^2}{\Big(f(z)\Big)^2} = -\frac{m}{z^2} - 2\beta - \sum_{k=1}^{\infty}\frac{1}{(z-\alpha_k)^2} < 0.$$

This shows that if $f(z)$ is not of the form $ce^{\alpha z}$ and if z is real but not a root of f, then

$$\Big(f'(z)\Big)^2 - f(z)f''(z) > 0. \tag{1}$$

By continuity

$$\Big(f'(z)\Big)^2 - f(z)f''(z) \geq 0 \tag{2}$$

for all real z with equality if and only if $f(z)$ is of the form $ce^{\alpha z}$ or z is a multiple root of f. Since the derivative of a function in $\mathcal{LP}$ is also in $\mathcal{LP}$, inequalities (1) and (2) apply to the derivatives of f. $\square$

Lemma 5 (Lemma 4.2 [1]) *Let $f \in \mathcal{LP}$ and let $h(z) = f(z) - \alpha^{-1} f'(z)$, where $\alpha \neq 0$ is real. If $h(z)$ has a zero of order $m \geq 2$ at x_0, then $f(z)$ has a zero of order at least $m+1$ at x_0. Consequently, if the zeros of f are simple, then the zeros of h are also simple.*

Proof. Since $h(z)$ has a zero of order m at x_0

$$0 = h^{(k)}(x_0) = f^{(k)}(x_0) - \alpha^{-1} f^{(k+1)}(x_0)$$

for $0 \leq k \leq m-1$. This implies that

$$f^{(k)}(x_0) = \alpha^k f(x_0)$$

for $0 \leq k \leq m$. Then for $1 \leq k \leq m-1$

$$\left(f^{(k)}(x_0)\right)^2 - f^{(k-1)}(x_0) f^{(k+1)}(x_0) = \left(\alpha^k f(x_0)\right)^2 - \left(\alpha^{k-1} f(x_0)\right)\left(\alpha^{k+1} f(x_0)\right) = 0.$$

Since $f, f', \ldots, f^{(m-1)}$ are not exponential functions (otherwise h could not have a zero of order m), the Laguerre Inequalities (Lemma 4) imply that

$$f^{(k)}(x_0) = 0$$

for $0 \leq k \leq m$. In other words, f has a zero of order at least $m+1$ at x_0. $\square$

Lemma 6 *Let $\phi_n(z) = \prod_{k=1}^{n} \left(1 - \frac{z}{\alpha_k}\right)$, where $\alpha_1, \alpha_2, \alpha_3, \ldots$ are real and nonzero, and let $f \in \mathcal{LP}$. Given $A > 0$ there exists N such that if $n \geq N$, then $\phi_n(D) f(z)$ has only simple zeros in the interval $(-A, A)$.*

Proof. Assume, to the contrary, that for some $A > 0$ there is a sequence $0 < n_1 < n_2 < n_3 < \cdots$ such that $\phi_{n_j}(D) f(z)$ has a zero x_j of multiplicity at least two in the interval $(-A, A)$. By Lemma 5, x_j is a zero of $f(z)$ of order at least $n_j + 2$. Since the sequence $n_j + 2$ is unbounded, $f(z)$ has zeros of arbitrarily large order in the finite interval $(-A, A)$. This is impossible since $f(z)$ is entire. $\square$

We will extend the previous lemma to show that if $\phi \in \mathcal{LP}$ and if ϕ has genus zero, then $\phi(D) f(z)$ has simple zeros. This is shown in Lemma 10. Lemmas 7 through 9 provide several technical results needed for the proof of Lemma 10.

Lemma 7 *Assume $f \in \mathcal{LP}$ is of the form*

$$f(z) = c z^m e^{\alpha z - \sigma z^2} \prod_{k=1}^{\infty} \left(1 - \frac{z}{\beta_k}\right) e^{z/\beta_k}.$$

and assume $\epsilon > 0$*. Then*

$$|f^{(n)}(z)| \leq n! A_\epsilon \left(\frac{2e(\sigma+\epsilon)}{n} \right)^{n/2}$$

for $|z| \leq R = \sqrt{\frac{n}{2(\sigma+\epsilon)}}$*, where* A_ϵ *is a constant depending on* ϵ*.*

Proof. As explained in [2, p. 13], the product

$$z^m e^{\alpha z} \prod_{k=1}^{\infty} \left(1 - \frac{z}{\beta_k}\right) e^{z/\beta_k}$$

(which lacks the term $e^{-\sigma z^2}$) is of order at most 2 and of minimal type. Thus $f(z)$ is of order 2 and normal type σ. Therefore, given $\epsilon > 0$ there exists A_ϵ such that

$$M_f(R) = \max_{|z| \leq R} |f(z)| < A_\epsilon \exp\Big((\sigma+\epsilon)R^2\Big)$$

for all R. By Cauchy's inequality, for $|z| \leq R$,

$$|f^{(n)}(z)| \leq \frac{n! M_f(R)}{R^n} \leq \frac{n! A_\epsilon \exp\Big((\sigma+\epsilon)R^2\Big)}{R^n}.$$

The last expression is minimized when $R = \sqrt{\frac{n}{2(\sigma+\epsilon)}}$. $\square$

Lemma 8 *For each* n *let*

$$\psi_n(z) = \prod_{k=n+1}^{\infty} \left(1 - \frac{z}{\alpha_k}\right),$$

where $\sum_{k=1}^{\infty} |\alpha_k|^{-1} < \infty$*. Then*

$$|\psi_n^{(k)}(0)| \leq k! \left(\frac{eB_n}{k}\right)^k,$$

where $B_n = \sum_{j=n+1}^{\infty} |\alpha_j|^{-1}$*.*

Proof. Let $M(R, \psi_n) = \max_{|z| \leq R} |\psi_n(z)|$. Taking the logarithm of the Weierstrass product for ψ_n gives

$$\log M(R, \psi_n) \leq \sum_{k=n+1}^{\infty} \log(1 + |R/\alpha_k|) \leq \sum_{k=n+1}^{\infty} |R/\alpha_k| = B_n R.$$

By Cauchy's inequality we obtain, for $|z| \leq R$,

$$|\psi_n^{(k)}(z)| \leq \frac{k! M(R, \psi_n)}{R^k} \leq \frac{k! \exp(B_n R)}{R^k}.$$

The last expression is minimized if $R = k/B_n$. $\square$

Lemma 9 *Let ψ_n be as in the previous lemma and let $f \in \mathcal{LP}$. Then $\psi_n(D)f(z)$ converges to $f(z)$ uniformly on compact sets.*

Proof. Let K be any compact subset of $\mathbb{C}$ and let $|z| < R$ for all $z \in K$. Then

$$\psi_n(D)f(z) = \sum_{k=0}^{\infty} \frac{\psi_n^{(k)}(0)}{k!} f^{(k)}(z).$$

Now let $\epsilon > 0$ as in Lemma 7. Then

$$|\psi_n(D)f(z) - f(z)| \le \sum_{1 \le k \le 2(\sigma+\epsilon)R^2} \frac{|\psi_n^{(k)}(0)|}{k!} |f^{(k)}(z)| + \sum_{k > 2(\sigma+\epsilon)R^2} \frac{|\psi_n^{(k)}(0)|}{k!} |f^{(k)}(z)|.$$

The reason for splitting the sum is that when $k > 2(\sigma+\epsilon)R^2$ the bound from Lemma 7 applies. Applying the bounds in Lemma 7 and 8 gives

$$|\psi_n(D)f(z) - f(z)| \le$$
$$\sum_{1 \le k \le 2(\sigma+\epsilon)R^2} \left(\frac{eB_n}{k}\right)^k \frac{k!M(R,f)}{R^k} + \sum_{k > 2(\sigma+\epsilon)R^2} \left(\frac{eB_n}{k}\right)^k k!A_\epsilon \left(\frac{2e(\sigma+\epsilon)}{k}\right)^{k/2}$$

The second summation converges by the root test from elementary calculus. Since $B_n \to 0$ as $n \to \infty$, the right hand side of the inequality can be made arbitrarily small when $|z| < R$ by taking n sufficiently large. This proves the uniform convergence. $\square$

Lemma 10 *Let $\phi(z) = \prod_{k=1}^{\infty}\left(1 - \frac{z}{\alpha_k}\right) \in \mathcal{LP}$ and let f be any function in $\mathcal{LP}$. Then $\phi(D)f(z)$ has only simple real zeros.*

Proof. Let A be any positive number. We will show that $\phi(D)f(z)$ has only simple zeros in the interval $(-A, A)$. We factor $\phi(z)$ as

$$\phi(z) = \phi_n(z)\theta_{n,m}(z)\psi_m(z)$$

where $1 \le n < m$ and where

$$\phi_n(z) = \prod_{k=1}^{n}\left(1 - \frac{z}{\alpha_k}\right), \quad \theta_{n,m}(z) = \prod_{k=n+1}^{m}\left(1 - \frac{z}{\alpha_k}\right), \quad \psi_m(z) = \prod_{k=m+1}^{\infty}\left(1 - \frac{z}{\alpha_k}\right).$$

Recalling that products in $\mathcal{LP}$ correspond to composition of differential operators we have

$$\phi(D)f(z) = \phi_n(D)\Big[\theta_{n,m}(D)\Big(\psi_m(D)f(z)\Big)\Big].$$

As the composition of these differential operators is commutative, the terms $\phi_n(D), \theta_{n,m}(D)$, and $\psi_m(D)$ can be written in any order. According to Lemma 6, there is an N such that $\phi_N(D)f(z)$ has only simple zeros in the interval $(-A, A)$. According to Lemma 9, $\psi_m(D)\Big(\phi_N(D)f(z)\Big)$ converges uniformly on compact sets to $\phi_N(D)f(z)$. By Hurwitz's Theorem the simple zeros of $\phi_N(D)f(z)$ are limit points of the zeros of $\psi_m(D)\Big(\phi_N(D)f(z)\Big)$. Thus, there exists an $M > N$ such that $\psi_M(D)\Big(\phi_N(D)f(z)\Big)$ has only simple zeros in the interval $(-A, A)$. Then by Lemma 5

$$\theta_{N,M}(D)\Big[\psi_M(D)\Big(\phi_N(D)f(z)\Big)\Big] = \phi(D)f(z)$$

has only simple zeros in the interval $(-A, A)$. Since A is arbitrary this proves the theorem. □

Lemma 11 *Let ϕ and f be in $\mathcal{LP}$. Write $\phi(z) = e^{-\alpha z^2}\phi_1(z)$ and $f(z) = e^{-\beta z^2}f_1(z)$, where ϕ_1 and f_1 have genus* 0 *or* 1 *and $\alpha, \beta \geq 0$. If $\alpha\beta < 1/4$ and ϕ has infinitely many zeros, then $\phi(D)f(z)$ has only simple real zeros.*

Proof. Since ϕ has infinitely many zeros, there is a subsequence $\{\alpha_k\}$ of zeros of ϕ such that $\sum_{k=1}^{\infty} |\alpha_k|^{-1} < \infty$. Write ϕ as

$$\phi(z) = \phi_0(z)\phi_2(z),$$

where $\phi_0(z) = \prod_{k=1}^{\infty}\Big(1 - \frac{z}{\alpha_k}\Big)$. Note that ϕ_0 has genus 0 and ϕ_2 has genus ≤ 2. By Lemma 2 $\phi_2(D)f(z)$ is in $\mathcal{LP}$. Then by Lemma 10

$$\phi(D)f(z) = \phi_0(D)\Big[\phi_2(D)f(z)\Big]$$

is in $\mathcal{LP}$ and has only simple zeros. □

This completes the proof of Theorem 1.

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