AN EQUIVALENCE FOR THE RIEMANN HYPOTHESIS IN TERMS OF ORTHOGONAL POLYNOMIALS

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Abstract. We construct a measure such that if $\{p_n(z)\}$ is the sequence of orthogonal polynomials relative to that measure, then the Riemann Hypothesis with simple zeros is true if and only if $\lim_{n\to\infty} \frac{p_{2n}(z)}{p_{2n}(0)} = \frac{\xi(1/2+iz)}{\xi(1/2)}$ where $\xi(s) = \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s)$ is the Riemann ξ-function.

1. Introduction

Let $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ be the Riemann zeta function. Riemann showed that $\zeta(s)$ has an analytic continuation to all s with the exception of a simple pole at $s = 1$. The Riemann ξ-function, defined as $\xi(s) = \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s)$, is an entire function satisfying $\xi(s) = \xi(1-s)$. The Riemann Hypothesis is the conjecture that all of the zeros of $\xi(s)$ lie on the 'critical line' which is the line with real part $1/2$. The Prime Number Theorem, proved independently by Hadamard and de la Vallée Poussin in 1896, is equivalent to the fact that all zeros of $\xi(s)$ lie in the critical strip $0 < \mathrm{Re}(s) < 1$. Let $M(T)$ denote the number of zeros in the critical strip with $0 < \mathrm{Im}(s) \le T$ that lie on the critical line. Hardy [6] proved that $M(T)$ tends to infinity as T tends to infinity. Hardy and Littlewood [7] showed that $M(T) > AT$ for some positive constant A. Selberg [12] proved that $M(T) > AT \log T$ for some positive constant A. Since the number $N(T)$ of zeros in the critical strip up to height T is known to be asymptotic to $\frac{T}{2\pi}\log(\frac{T}{2\pi})$, Selberg showed that a positive proportion of the zeros are on the critical line. Extensive numerical calculations, such as [2, 3, 8, 9, 10, 11], have supported the Riemann Hypothesis. The numerical calculations have supported the stronger conjecture that, in addition to lying on the critical line, the zeros of $\xi(s)$ are *simple*. In this paper we show that the Riemann Hypothesis with simple zeros is equivalent to the existence of a certain family of orthogonal polynomials $\{p_n(z)\}$ such that $\lim_{n\to\infty} \frac{p_{2n}(z)}{p_{2n}(0)} = \frac{\xi(1/2+iz)}{\xi(1/2)}$.

We will now describe the main result of this paper. We will define a step function F related to the zeros of $\xi(s)$. Let $\Xi(z) = \xi(1/2 + iz)$. Then the zeros of $\Xi(z)$ lie in the strip $-1/2 < \mathrm{Im}(z) < 1/2$, $\Xi(z)$ is real for real z, $\Xi(z) = \Xi(-z)$, and any non-real zeros of $\Xi(z)$ occur in complex conjugate pairs. For $z = x + iy$ in the region $x \ge 0$, $-1/2 \le y \le 1/2$, let $f(z)$ be analytic and satisfy:

$$f(z) \text{ is real for real } z, \tag{1}$$

$$\mathrm{Re}\big(f(z)\big) > 0, \tag{2}$$

$$|f(x+iy)| < e^{-cx} \tag{3}$$

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where c is a positive constant. For $T \geq 0$ let

$$F(T) = \frac{1}{2\pi i} \int_{\gamma_T} \frac{\Xi'(z)}{\Xi(z)} f(z)\, dz \tag{4}$$

where γ_T is the positively oriented boundary of the region $0 \leq x \leq T$, $-1/2 \leq y \leq 1/2$. Label the zeros of $\Xi(z)$ in the region $x > 0$, $0 \leq y < 1/2$ as $\alpha_k + i\beta_k$ where $\alpha_k \leq \alpha_{k+1}$. If T is not equal to any α_k, $F(T)$ may be represented as the finite sum

$$F(T) = \sum_{\substack{\alpha_k < T \\ \beta_k = 0}} f(\alpha_k) + \sum_{\substack{\alpha_k < T \\ \beta_k > 0}} \big\{ f(\alpha_k + i\beta_k) + f(\alpha_k - i\beta_k) \big\}.$$

For $T < 0$ let $F(T) = -F(T)$. If $f(z)$ were replaced by 1, $F(T)$ would be the number of zeros in the critical strip up to height T. However, we imposed the restriction in inequality (3) to guarantee the existence of certain integrals.

For polynomials $p(x)$ and $q(x)$ with real coefficients we define an inner product by the Riemann-Stieltjes integral

$$\langle p(x), q(x) \rangle = \int_{-\infty}^{\infty} p(x) q(x) dF(x).$$

Applying the Gram-Schmidt orthogonalization process to the polynomials $1, x, x^2, \ldots$ produces an orthogonal family of polynomials $\{p_n(x)\}$ where the degree of $p_n(x)$ is n. In this case, $p_{2n}(x)$ is an even function while $p_{2n+1}(x)$ is an odd function.

Then we have:

Theorem 1. *The Riemann hypothesis with simple zeros is true if and only if*

$$\lim_{n \to \infty} \frac{p_{2n}(z)}{p_{2n}(0)} = \frac{\xi(1/2 + iz)}{\xi(1/2)}$$

for every $z \in \mathbb{C}$.

We note that the proof shows that $\lim_{n\to\infty} \frac{p_{2n}(z)}{p_{2n}(0)} = \lim_{n\to\infty} \frac{p_{2n+1}(z)}{z\, p'_{2n+1}(0)}$. Thus the theorem could be stated in terms of the odd degree polynomials $p_{2n+1}(z)$ as well.

The proof of Theorem 1 is presented in §3.

2. A few facts about orthogonal polynomials

In this section we will recall several facts from the general theory of orthogonal polynomials that will be needed in the proof of Theorem 1. For the basic theory, we refer the reader to the books by Szëgo [14] and Chihara [4]. Our review is based on these two works but especially on [4].

A bounded non-decreasing function ψ is called a *distribution function* if its moments

$$\mu_n = \int_{-\infty}^{\infty} x^n d\psi(x) \tag{5}$$

exist for $n = 0, 1, 2, \ldots$. Two distribution functions ψ_1 and ψ_2 are *substantially equal* if and only if there is a constant K such that $\psi_1(x) = \psi_2(x) + K$ at all common points of continuity. The *spectrum* of ψ is the set

$$\mathfrak{S}(\psi) = \{x \,|\, \psi(x+\delta) - \psi(x-\delta) > 0 \quad \text{for all } \delta > 0\}.$$

If $\mathfrak{S}(\psi)$ is infinite, then the expression

$$\langle p(x), q(x)\rangle = \int_{-\infty}^{\infty} p(x)q(x)d\psi(x)$$

defines an inner product on the space of polynomials with real coefficients. Using this inner product we orthogonalize the set of non-negative powers of x to create a family $\{p_n(x)\}$ of orthogonal polynomials with real coefficients using the Gram-Schmidt procedure:

$$p_0(x) = 1,$$
$$p_n(x) = x^n - \sum_{k=0}^{n-1} \frac{\langle x^n, p_k\rangle}{\langle p_k, p_k\rangle} p_k(x) \quad \text{for } k \geq 1.$$

Lemma 2.1. ([14, Thm. 3.3.1] *or* [4, Thm. I.5.2].) *The zeros of $p_n(x)$ are real and simple for each $n \geq 1$.*

We will label the zeros of $p_n(x)$ as $y_{n1} < y_{n2} < \cdots < y_{nn}$.

Lemma 2.2. ([14, Thm. 3.3.3] *or* [4, Thm. I.5.3].) *The zeros of $p_n(x)$ and $p_{n+1}(x)$ interlace. That is,*

$$y_{n+1,i} < y_{ni} < y_{n+1,i+1}, \qquad i = 1, 2, \ldots, n.$$

Furthermore, between any two zeros of $p_n(x)$ there is at least one zero of $p_m(x)$ for $m > n$.

Using the moments from equation (5) we define a *moment functional* on the space of polynomials by

$$\mathcal{L}[p(x)] = \int_{-\infty}^{\infty} p(x)d\psi(x) = \sum_{k=0}^{n} c_k\mu_k$$

where $p(x) = c_0 + c_1x + \cdots + c_nx^n$.

Lemma 2.3. ([14, Thm. 3.4.1] *or* [4, Thm. I.6.1].) *There are numbers A_{n1}, $A_{n2}, \ldots, A_{nn}$ such that for every polynomial $\pi(x)$ of degree at most $2n-1$*

$$\mathcal{L}[\pi(x)] = \sum_{k=1}^{n} A_{nk}\pi(y_{nk}).$$

The numbers A_{nk} are all positive and $A_{n1} + \cdots + A_{nn} = \mu_0$.

The equation in Lemma 2.3 is called the *Gauss quadrature formula*. The numbers A_{nk} are sometimes called *Christoffel numbers*.

The zeros of the polynomials $\{p_n(x)\}$ are strongly related to the spectrum $\mathfrak{S}(\psi)$. Let

$$\psi_n(x) = \begin{cases} 0 & \text{if } x < y_{n1}, \\ A_{n1} + \cdots + A_{np} & \text{if } y_{np} \leq x < y_{n,p+1} \text{ where } 1 \leq p < n, \\ \mu_0 & \text{if } x \geq y_{nn}. \end{cases} \tag{6}$$

Lemma 2.4. ([4, Thm. II.3.1].) *There is a subsequence of $\{\psi_n\}$ that converges on $(-\infty, \infty)$ to a distribution function η which has a infinite spectrum and such that $\mu_n = \int_{-\infty}^{\infty} x^n d\psi(x) = \int_{-\infty}^{\infty} x^n d\eta(x)$.*

It is *not* generally true that η is substantially equal to ψ. Distribution functions, such as η, that are subsequential limits of $\{\psi_n\}$ are called *natural representatives* of the moment functional $\mathcal{L}$.

Lemma 2.5. ([14, Thm. 3.41.2] *or* [4, Thm. II.4.1].) *The open interval* $(y_{ni}, y_{n,i+1})$ *contains at least one spectral point of the function* ψ.

Lemma 2.6. ([4, Thm. II.4.3].) *Let* η *be a natural representative of* $\mathcal{L}$ *and let* $s \in \mathfrak{S}(\eta)$. *Then every neighborhood of* s *contains a zero of* $p_n(x)$ *for infinitely many values of* n.

Given a list of moments $\{\mu_n\}_{n=0}^{\infty}$, the *Hamburger moment problem* consists of classifying the distribution functions ϕ such that

$$\mu_n = \int_{-\infty}^{\infty} x^n d\phi(x), \qquad n = 0, 1, 2, \dots$$

If all solutions ϕ of the Hamburger moment problem are substantially equal, we say the moment problem is *determined*.

Carleman gave a sufficient (but not necessary) condition for a moment problem to be determined.

Lemma 2.7. ([13, Thm. 1.11] *or* [1, p. 85].) *The moment problem* $\mu_n = \int_{-\infty}^{\infty} x^n d\psi(x)$ *is determined if*

$$\sum_{n=1}^{\infty} \mu_{2n}^{-1/2n} = \infty.$$

The most crucial part of the proof of Theorem 1 will involve showing that the Hamburger moment problem for the distribution function F, defined in equation (4), is determined. We will now proceed with the proof.

3. Proof of Theorem 1

We begin by showing that the expression

$$\langle p(x), q(x) \rangle = \int_{-\infty}^{\infty} p(x)q(x)dF(x) \tag{7}$$

defines an inner product on the space of polynomials with real coefficients.

Lemma 3.1. *The nth moments*

$$\mu_n = \int_{-\infty}^{\infty} x^n dF$$

exist, and equation (7) *defines an inner product on the space of polynomials with real coefficients.*

Proof. Label the zeros of $\xi(1/2 + iz)$ in the region $\{x + iy \,|\, x > 0,\ 0 \leq y < 1/2\}$ as $\alpha_k + i\beta_k$ where $\alpha_k \leq \alpha_{k+1}$ for $k \geq 1$. If $\alpha_k + i\beta_k$ is a root, so is $\alpha_k - i\beta_k$. Also recall from (3) that $|f(x + iy)| < \exp(-cx)$ when $x > 0$ and $-1/2 < y < 1/2$. Then

$$\begin{aligned} \int_0^{\infty} x^n dF &= \sum_{\substack{k \\ \beta_k = 0}} \alpha_k^n f(\alpha_k) + \sum_{\substack{k \\ \beta_k \neq 0}} \alpha_k^n \big(f(\alpha_k + i\beta_k) + f(\alpha_k - i\beta_k)\big) \\ &\leq \sum_{k=1}^{\infty} \alpha_k^n e^{-c\alpha_k} + 2\sum_{k=1}^{\infty} \alpha_k^n e^{-c\alpha_k} = 3\sum_{k=1}^{\infty} \alpha_k^n e^{-c\alpha_k}. \end{aligned} \tag{8}$$

We need to know the approximate size of α_k. Recall that the number $N(T)$ of zeros of $\xi(z)$ in the critical strip up to height T is known [15, p. 214] to satisfy

$$N(T) \sim \frac{T}{2\pi} \log\left(\frac{T}{2\pi}\right).$$

It follows that if the zeros in the critical strip with $\mathrm{Im}(z) > 0$ are labelled as $\rho_k + it_k$ with $t_{k+1} \geq t_k$, then

$$t_k \sim \frac{2\pi k}{\log k}. \tag{9}$$

By equation (9) there exist positive constants A and B such that

$$A\frac{k}{\log k} < \alpha_{k-1} < B\frac{k}{\log k} \tag{10}$$

for $k \geq 2$. Combining inequalities (8) and (10) gives

$$\int_0^\infty x^n dF \leq 3B^n \sum_{k=2}^\infty \left(\frac{k}{\log k}\right)^n \exp\left(-cA\frac{k}{\log k}\right).$$

The sum clearly converges. This can be seen, for example, by using the limit comparison test from elementary calculus with the convergent series $\sum k^{-2}$. Because $F(T) = -F(-T)$, $\mu_n = 0$ for odd n. When n is even

$$\mu_n = \int_{-\infty}^\infty x^n dF = 2\int_0^\infty x^n dF \leq 6B^n \sum_{k=2}^\infty \left(\frac{k}{\log k}\right)^n \exp\left(-cA\frac{k}{\log k}\right). \tag{11}$$

This shows that the moments $\mu_n = \int_{-\infty}^\infty x^n dF$ exist for $n \geq 0$. Thus the expression $\langle p(x), q(x)\rangle$, defined by equation (7), exists for any real polynomials $p(x)$ and $q(x)$. The bilinearity is apparent. Because the measure dF has infinite support $\langle p(x), p(x)\rangle > 0$ unless $p(x) = 0$. Therefore, the expression $\langle p(x), q(x)\rangle$ defines an inner product on the space of polynomials with real coefficients. □

Lemma 3.2. *The Hamburger moment problem for the moments of the distribution function F,*

$$\mu_n = \int_{-\infty}^\infty x^n dF,$$

is determined.

Proof. By extending the proof of the previous lemma we will obtain a sufficiently good upper bound on μ_n to apply Carleman's criterion (Lemma 2.7). We begin by estimating the summation in inequality (11). Let

$$S(n) = \sum_{k=2}^\infty \left(\frac{k}{\log k}\right)^n \exp\left(-cA\frac{k}{\log k}\right).$$

Split the sum into two parts:

$$S(n) = \underbrace{\sum_{2\leq k\leq M+1} \left(\frac{k}{\log k}\right)^n \exp\left(-cA\frac{k}{\log k}\right)}_{S_1(n)} + \underbrace{\sum_{k>M+1} \left(\frac{k}{\log k}\right)^n \exp\left(-cA\frac{k}{\log k}\right)}_{S_2(n)}.$$

We will determine bounds for $S_1(n)$ and $S_2(n)$. A careful choice of M will lead to a bound on $S_2(n)$ that is much smaller than the bound on $S_1(n)$.

By elementary calculus the function $(\frac{k}{\log k})^n \exp(-cA\frac{k}{\log k})$ has a maximum of $(\frac{n}{ecA})^n$ when $\frac{k}{\log k} = \frac{n}{cA}$. This gives a bound on $S_1(n)$:

$$S_1(n) \leq M \left(\frac{n}{ecA}\right)^n. \tag{12}$$

Now assume that M is sufficiently large such that the following three conditions hold:

$$\frac{k}{\log k} > \frac{n}{cA} \quad \text{for } k \geq M, \tag{13}$$

$$\left(\frac{k}{\log k}\right)\left(\frac{\log k - 1}{\log^2 k}\right) > 1 \quad \text{for } k \geq M, \tag{14}$$

$$\frac{M}{\log M} > \left(\frac{2(n+1)}{cA}\right)^2. \tag{15}$$

Condition (13) ensures that the function $(\frac{k}{\log k})^n \exp(-cA\frac{k}{\log k})$ decreases for $k \geq M$. The reasons for assuming conditions (14) and (15) will become apparent in the following calculation:

$$\begin{aligned} S_2(n) &= \sum_{k > M+1} \left(\frac{k}{\log k}\right)^n \exp\left(-cA\frac{k}{\log k}\right) \\ &< \int_M^\infty \left(\frac{k}{\log k}\right)^n \exp\left(-cA\frac{k}{\log k}\right) dk && \text{by (13)}, \\ &< \int_M^\infty \left(\frac{k}{\log k}\right)^{n+1} \exp\left(-cA\frac{k}{\log k}\right)\left(\frac{\ln k - 1}{\ln^2 k}\right) dk && \text{by (14)}, \\ &= \int_{M/\log M}^\infty w^{n+1} \exp(-cAw)\, dw. \end{aligned}$$

For any positive α and w, $w < \frac{\exp(\alpha w)}{\alpha}$. Setting $\alpha = \frac{cA}{2(n+1)}$ gives

$$S_2(n) < \left(\frac{2(n+1)}{cA}\right)^{n+1} \int_{M/\log M}^\infty \exp\left(-\frac{cAw}{2}\right) dw = \frac{2}{cA}\left(\frac{\frac{2(n+1)}{cA}}{\exp(\frac{cA}{2(n+1)}\frac{M}{\log M})}\right)^{n+1}.$$

By condition (15), $\frac{2(n+1)}{cA} < \exp(\frac{cA}{2(n+1)}\frac{M}{\log M})$. Thus

$$S_2(n) < \frac{2}{cA}. \tag{16}$$

Combining inequalities (12) and (16) gives

$$S(n) = S_1(n) + S_2(n) < M\left(\frac{n}{ecA}\right)^n + \frac{2}{cA}.$$

Let $M = \kappa^n$ where $\kappa > 1$. As soon as n is sufficiently large conditions (13), (14), and (15) hold. So, for sufficiently large even n,

$$\begin{aligned}\mu_n^{1/n} &\le \left(6B^n S(n)\right)^{1/n} < \left(6B^n\left(\left(\frac{\kappa n}{ecA}\right)^n + \frac{2}{cA}\right)\right)^{1/n}\\ &< \left(12B^n\left(\frac{\kappa n}{ecA}\right)^n\right)^{1/n} = 12^{1/n}\left(\frac{B\kappa}{ecA}\right)n\\ &< \left(\frac{2B\kappa}{ecA}\right)n.\end{aligned}$$

Consequently

$$\sum_{n=0}^{\infty} \mu_{2n}^{-1/2n} = \infty,$$

and by Carleman's criterion (Lemma 2.7) it follows that the Hamburger moment problem $\mu_n = \int_{-\infty}^{\infty} x^n dF(x)$ is determined. □

Let $\{p_n(x)\}$ be the family of orthogonal polynomials obtained from the inner product in Lemma 3.1 by orthogonalizing the set of non-negative powers of x according the the Gram-Schmidt procedure. Because $\mu_{2k+1} = 0$ and $\mu_{2k} > 0$ for each k it follows that $p_{2n+1}(x)$ is an odd function while $p_{2n}(x)$ is an even function.

The spectrum (defined in §2) of F consists of all α_k such that $\alpha_k + i\beta_k$ is a zero of $\xi(1/2 + iz)$. We will label the positive values in $\mathfrak{S}(F)$ as

$$a_1 < a_2 < a_3 < \cdots .$$

It was known, as early as Riemann [5, p. 159], that $a_1 \approx 14.134$. Denote the n positive zeros of $p_{2n}(x)$ as

$$x_{2n,1} < x_{2n,2} < \cdots < x_{2n,n}.$$

Similarly, denote the n positive zeros of $p_{2n+1}(x)$ as

$$x_{2n+1,1} < x_{2n+1,2} < \cdots < x_{2n+1,n}.$$

Thus, we may write

$$\frac{p_{2n}(z)}{p_{2n}(0)} = \prod_{k=1}^{n}\left(1 - \frac{z^2}{x_{2n,k}^2}\right) \quad \text{and} \quad \frac{p_{2n+1}(z)}{z\, p'_{2n+1}(0)} = \prod_{k=1}^{n}\left(1 - \frac{z^2}{x_{2n+1,k}^2}\right). \tag{17}$$

In Lemmas 3.3 and 3.4 we will show that $a_k = \lim_{n\to\infty} x_{nk}$.

Lemma 3.3. *$a_k < x_{mk} < x_{nk}$ when $m > n$. Hence, $a_k \le \lim_{n\to\infty} x_{nk}$.*

Proof. The spectral points of F are the numbers $\pm a_k$ for $k = 1, 2, 3, \ldots$. By Lemma 2.5 if n is odd, the open interval $(0, x_{n1})$ contains a_1 because a_1 is the smallest positive spectral point. If n is even, the open interval $(-x_{n1}, x_{n1})$ contains a_1. In either case, $a_1 < x_{n1}$. Similarly, each of the open intervals $(x_{n,k-1}, x_{nk})$ for $2 \le k \le \lfloor n/2 \rfloor$ contains a spectral point. This forces $a_k < x_{nk}$. From the interlacing property of zeros in Lemma 2.2 it is immediate that

$$0 < x_{n+1,1} < x_{n,1} < \cdots < x_{n+1,n} < x_{n,n} < x_{n+1,n+1} \tag{18}$$

whether n is even or odd. Hence, $a_k < x_{mk} < x_{nk}$ for when $m > n$. □

Lemma 3.4. $a_k = \lim_{n\to\infty} x_{nk}$.

Proof. By Lemma 2.4 there is a subsequence of the functions F_n, defined in equation (6), that converges to a distribution function η such that

$$\mu_n = \int_{-\infty}^{\infty} x^n d\eta(x), \quad n = 0, 1, 2, \dots$$

In Lemma 3.2 it was established that the Hamburger moment problem

$$\mu_n = \int_{-\infty}^{\infty} x^n dF(x), \quad n = 0, 1, 2, \dots$$

is determined. Therefore, F and η are substantially equal and they have the *same* spectrum. Let a_k be any one of the positive spectral points of F or η. By Lemma 2.6 every neighborhood of a_k contains a zero of $p_n(x)$ for infinitely many n. Let $\delta_1 > 0$ be small enough so that the only spectral point of F in $(a_1 - \delta_1, a_1 + \delta_1)$ is a_1. By Lemma 3.3 the only root of $p_n(x)$ that potentially could be in that neighborhood is x_{n1}. Since infinitely many values of the bounded decreasing sequence $\{x_{n1}\}$ lie in that neighborhood of a_1, $\lim_{n\to\infty} x_{n1} = a_1$. Suppose, by way of induction, that $\lim_{n\to\infty} x_{nr} = a_r$ for $1 \le r < k$. Choose $\delta_k > 0$ small enough so that the only spectral point of F in $(a_k - \delta_k, a_k + \delta_k)$ is a_k. Again by Lemma 3.3 the roots x_{nj} for $j > k$ cannot be in the neighborhood of a_k since $a_k < a_{k+1} < x_{nj}$. By the induction hypothesis only finitely many roots x_{nj} with $j < k$ can be in the neighborhood. Since the neighborhood contains infinitely many roots the only possibility is that x_{nk} is in the neighborhood for infinitely many n. Thus $\lim_{n\to\infty} x_{nk} = a_k$. □

Lemma 3.5. *The sequences of polynomials*

$$\frac{p_{2n}(z)}{p_{2n}(0)} \qquad \text{and} \qquad \frac{p_{2n+1}(z)}{z\, p'_{2n+1}(0)}$$

converge uniformly on compact sets to the entire function with simple real zeros corresponding to the real parts of zeros of $\xi(1/2 + iz)$*. Thus, for all* $z \in \mathbb{C}$,

$$\lim_{n\to\infty} \frac{p_{2n}(z)}{p_{2n}(0)} = \lim_{n\to\infty} \frac{p_{2n+1}(z)}{z\, p'_{2n+1}(0)} = \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{a_k^2}\right).$$

Proof. Let $\epsilon > 0$ be given. Let K be any compact subset of $\mathbb{C}$. Choose R so that $|z| < R$ for every $z \in K$. Define M_R to the be positive constant

$$M_R = \prod_{k=1}^{\infty} \left(1 + \frac{R^2}{a_k^2}\right).$$

Because $\xi(1/2 + iz)$ is an entire function of order one [15, Thm. 2.12], $\sum_{k=1}^{\infty} a_k^{-2} < \infty$. Consequently, M_R is finite. For $z \in K$,

$$\left|\frac{p_{2n}(z)}{p_{2n}(0)}\right| = \left|\prod_{k=1}^{n} \left(1 - \frac{z^2}{x_{2n,k}^2}\right)\right| \le \prod_{k=1}^{n} \left(1 + \frac{|z|^2}{x_{2n,k}^2}\right) \le \prod_{k=1}^{n} \left(1 + \frac{R^2}{a_k^2}\right) \le M_R.$$

Now choose N at least large enough so that $a_k > R$ when $k > N$. For $n > N$, define $\alpha(z)$ and $\beta(z)$ by

$$1 + \alpha(z) = \prod_{k=N+1}^{n} \left(1 - \frac{z^2}{x_{2n,k}^2}\right) \qquad \text{and} \qquad 1 + \beta(z) = \prod_{k=N+1}^{n} \left(1 - \frac{z^2}{a_k^2}\right).$$

Since $1 - R^2/a_k^2 < |1 - z^2/x_{2n,k}^2| < 1 + R^2/a_k^2$ we obtain

$$\prod_{k=N+1}^{\infty} \left(1 - R^2/a_k^2\right) < |1 + \alpha(z)| < \prod_{k=N+1}^{\infty} \left(1 + R^2/a_k^2\right).$$

Similarly,

$$\prod_{k=N+1}^{\infty} \left(1 - R^2/a_k^2\right) < |1 + \beta(z)| < \prod_{k=N+1}^{\infty} \left(1 + R^2/a_k^2\right).$$

Since $\lim_{N\to\infty} \prod_{k=N+1}^{\infty} \left(1 - R^2/a_k^2\right) = 1$ and $\lim_{N\to\infty} \prod_{k=N+1}^{\infty} \left(1 - R^2/a_k^2\right) = 1$ we may choose N large enough so that

$$|\alpha(z)| < \frac{\epsilon}{M_R} \qquad \text{and} \qquad |\beta(z)| < \frac{\epsilon}{M_R}.$$

Choose $N_1 > N$ large enough so that, if $n > N_1$ and $z \in K$,

$$\left| \prod_{k=1}^{N} \left(1 - \frac{z^2}{x_{2n,k}^2}\right) - \prod_{k=1}^{N} \left(1 - \frac{z^2}{a_k^2}\right) \right| < \epsilon.$$

Let $n > N_1$. Then

$$\begin{aligned}
&\left| \frac{p_{2n}(z)}{p_{2n}(0)} - \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{a_k^2}\right) \right| = \left| \prod_{k=1}^{n} \left(1 - \frac{z^2}{x_{2n,k}^2}\right) - \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{a_k^2}\right) \right| \\
&= \left| \left(1 + \alpha(z)\right) \prod_{k=1}^{N} \left(1 - \frac{z^2}{x_{2n,k}^2}\right) - \left(1 + \beta(z)\right) \prod_{k=1}^{N} \left(1 - \frac{z^2}{a_k^2}\right) \right| \\
&\le \left| \prod_{k=1}^{N} \left(1 - \frac{z^2}{x_{2n,k}^2}\right) - \prod_{k=1}^{N} \left(1 - \frac{z^2}{a_k^2}\right) \right| \\
&\qquad + |\alpha(z)| \left| \prod_{k=1}^{N} \left(1 - \frac{z^2}{x_{2n,k}^2}\right) \right| + |\beta(z)| \left| \prod_{k=1}^{N} \left(1 - \frac{z^2}{a_k^2}\right) \right| \\
&\le \epsilon + \frac{\epsilon}{M_R} \cdot M_R + \frac{\epsilon}{M_R} \cdot M_R = 3\epsilon.
\end{aligned}$$

This shows that $\frac{p_{2n}(z)}{p_{2n}(0)}$ converges to $\prod_{k=1}^{\infty}(1 - z^2/a_k^2)$ uniformly on compact subsets of $\mathbb{C}$ as n tends to infinity. The same argument, with $2n$ replaced by $2n+1$, shows that the sequence $\frac{p_{2n+1}(z)}{z\, p_{2n+1}'(0)}$ converges uniformly on compact sets to the same entire function. □

By Lemma 3.5

$$\lim_{n\to\infty} \frac{p_{2n}(z)}{p_{2n}(0)} = \lim_{n\to\infty} \frac{p_{2n+1}(z)}{z\, p_{2n+1}'(0)} = \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{a_k^2}\right) = \frac{\xi(1/2 + iz)}{\xi(1/2)}$$

if and only if $\xi(1/2 + iz)$ has simple real zeros. This completes the proof of Theorem 1.

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