THE THREE POINT STEINER PROBLEM ON THE FLAT TORUS: THE MINIMAL LUNE CASE

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Abstract. We show how to identify the minimal path network connecting three fixed points on the flat torus under certain conditions. In particular, let T be a torus covered by a rectangular fundamental domain of dimension $l \times w$. Let A, B, and C be points in T. Associated with A and B is a minimal lune region, $L(A, B)$, which will be described in our paper. If $AB \leq \frac{\sqrt{3}}{4}\sqrt{l^2 + w^2}$ and C is contained in $L(A, B)$, then the minimal path network is the union of the segments connecting A to C and C to B, which are contained in $L(A, B)$.

1. Introduction

In this paper, we are going to solve the three-point Steiner problem for a special case on the flat torus. The Steiner problem asks to find the minimal path network connecting n specified points. Given three points in the plane, A, B, and C, if one of the angles of the $\triangle ABC$ is greater than $120°$, then the minimal path network in the plane is constructed by connecting the two edges that are adjacent to that angle. If all of the angles of the triangle are less than $120°$, the minimal path network contains an additional point, called a Steiner point. There are algorithms that can solve any given n point Steiner problem in the plane. The first such algorithm is Melzak's algorithm.

The problem on the torus is significantly harder than the Euclidean problem. In the Euclidean plane there is only one straight line that connects two points, but on the torus there are infinitely many straight lines that connect two points. This problem is important because many people have worked on it. It is also groundbreaking work for similar problems, on other surfaces.

2. Preliminaries

We can relate the Steiner problem on the flat torus to the Steiner problem on the Euclidean plane. The flat torus has a rectangular fundamental domain in the Euclidean plane that acts as a covering space for the torus. By taking a rectangular fundamental domain in the Euclidean plane and "sewing" it's parallel edges together, we can represent the flat torus.

Then because on a torus we can create a path network over the "seams", to represent the torus in the Euclidean plane, we need to tile the Euclidean plane with fundamental domains. Let the dimensions of the fundamental domain be $l \times w$. Let a fundamental domain be denoted by $T_{m,n}$, where m denotes the horizontal position and n denotes the vertical position of the fundamental domain. Let the original fundamental domain the fundamental domain defined by the positive x and y axis with the origin as a corner be denoted $T_{0,0}$. Let the fundamental domain directly above will be $T_{0,1}$, the fundamental domain directly to the right will be $T_{1,0}$, and the fundamental domain that is above $T_{1,0}$ will be $T_{1,1}$.

So, given three points on the torus, each fundamental domain, $T_{m,n}$, will have three points to represent these points. Connecting points in different domains represents the path network crossing over a "seam". If a point A is located at (x, y), then the translates of A will be at $(x + ml, y + nw)$ where l is the length of the fundamental domain along the x-axis, w is the width along the y-axis, and m and n are indicators as to which fundamental domain the translate is in. For example, if the translate is in $T_{1,0}$, then $m = 1$ and $n = 0$. Given points A, B, C on the torus, we will refer to the translates of these points as A', B', and C'.

2.1. **The Three Point Steiner Problem in Euclidean Space.** Given three points A, B, and C in the Euclidean plane, if one of the angles of the $\triangle ABC$ is at least 120° then the minimal path network is the connection of the two edges forming the obtuse angle, called the degenerate tree. If the angles are less than 120°, then an extra vertex S is added to form the minimal path network, and this type of tree is called a Steiner tree.

Given A and B, a third point C can lie in one of five regions. Two of these regions are Steiner regions, where all the angles in $\triangle ABC$ are less than 120°. The length of the tree can easily be found in these regions. Two equilateral triangles are constructed with common edge $\overline{AB}$ denoting the two third points of the equilateral triangles as E-points. The length of the Steiner tree is the length of the Simpson line, which is the line from C to the E-point on the opposite side of $\overline{AB}$. The other three regions are degenerate; two of them connect A and B directly and then connects C to one of the endpoints, and the last connects A to C and C to B. This last region is called the lune of A and B, denoted by $L(A, B)$.

2.2. **Euclidean Space vs. the Flat Torus.** The Steiner problem is more difficult than the Steiner problem in Euclidean Space because on the torus there are an infinite number of straight lines that connect to points on the torus, whereas in Euclidean space there is only one straight line between two points. To represent the infinite number of lines between two points on the torus, we have tiled the Euclidean plane with fundamental domains that contain translates of all the points on the torus. If A and B are on the torus, then if you choose a point in the Euclidean plane that represents A, then each different segment between A and a representation of B represents a different way to connect A and B on the torus.

This unique feature makes the Steiner problem on the torus more difficult than in Euclidean Space. Fortunately, Keith Penrod has proved that for three points on the flat torus, the minimal path network must be contained in one rectangular fundamental domain. While this result reduces the number of possible cases, the problem on the torus is still more difficult because for three points, there are still twelve translates of each point to consider in the three-point problem.

3. Overview

We show how to identify the minimal path network connecting three fixed points on the flat torus under certain conditions. In particular, let T be a torus covered by a rectangular fundamental domain of dimension $l \times w$. Let A, B, and C, be points in T. Associated with A and B is a minimal lune region, $L(A, B)$. If $AB \leq \frac{\sqrt{3}}{4}\sqrt{l^2 + w^2}$ and C is contained in $L(A, B)$, then the minimal path network is the union of the segments connecting A to C and C to B, which are contained in $L(A, B)$.

Given two fixed points A and B, and one movable point, C, the three regions of interest can be found. The lune region is the space where $\angle ACB$ is greater than or equal to 120°. The degenerate regions are where either $\angle CAB$ or $\angle ABC$ is greater than or equal to 120°. The full Steiner regions are above and below the lune, where the $\angle ACB$ is less than 120°.

With three steps we can find when the lune path network is minimal:

Step 1. **Maximal Lune Network** Finding the maximal possible length for a minimal path network in $L(A, B)$.
Step 2. **Degenerate Case** Finding the minimal path network in the degenerate regions with the point in this region being a translate of one in the lune.
Step 3. **Steiner Case** Finding the minimal path network in the Steiner regions with the point in this region being a translate of one in the lune.

By using these results, we find a condition on A and B such that if it is met, then the minimal path network will be contained in $L(A, B)$.

4. Maximum Path Network Length For a Tree in the Lune

We can construct a lune by circumscribing two circles, one around each of the two equilateral triangles that have $\overline{AB}$ as a common edge. The centers of these circles we define as lune foci. These two circles intersect at points A and B. The boundary of the lune is defined by the two arcs between A and B. Let $L(A, B)$ denote the lune of A and B, and let $\partial L(A, B)$ signify the boundary of the lune.

Because all of the angles of an equilateral triangle are 60°, the arc between A and B is 120°. This implies that all points on $\partial L(A, B)$ form a 120° angle with A and B. Given a point C in the lune, the shortest tree to connect A, B, and C is to connect A to C and C to B.

Theorem 4.1. *Suppose A and B are points in the Euclidean plane. Then amongst all the points C of $L(A, B)$, the sum of the distances $AC + CB$ is maximized at the points equidistant from A and B on $\partial L(A, B)$.*

Proof. Let M be a point in the $L(A, B)$ that forms a tree with maximal length. Note that because of symmetry, there would exist two points with this property. Also because of symmetry we can find one maximal tree, and the other will be a reflection.

First we know that the M that forms the maximal tree cannot be in $L(A, B)$. Otherwise, there would exist an X on the line perpendicular to $\overline{AB}$ through the point M and between M and $\partial L(A, B)$. Let the point D define the intersection of the perpendicular line and $\overline{AB}$. Then by the Pythagorean Theorem, we know that

$$AD^2 + XD^2 = AX^2$$
$$AD^2 + DM^2 = AM^2$$

then

$$AX^2 - XD^2 = AC^2 - MD^2$$

and because we know that $XD > MD$,

$$AX^2 - XD^2 > AM^2 - XD^2$$
$$AX^2 > AM^2$$

Therefore $AX > AM$. Similarly $BX > BM$. And then $AX + BX > AM + BM$, contradicting that M defines a maximum length tree in the lune.

Therefore M and M^*, the reflection of M about AB, must be on $\partial L(A, B)$. Let O be the lune focus of $L(A, B)$ corresponding to the arc containing M, and O^* to the arc containing M^* respectively. Let each of these circles have radius r.

Without loss of generality, consider the point M on $\partial L(A, B)$. We know that the $\angle AOB$ is 120°. Let θ be the angle $\angle BOM$, let α be $\angle OBM$, and by the law of sines, $\angle OMB$ is equal to α. Then $\angle AOM$ is 120 - θ, $\angle OAM$ is 120 - α, and $\angle AMO$ is 120 - α.

(Picture)

If we bisect the angles θ and 120 - θ, then

$$AM = 2r \sin \frac{\theta}{2}$$
$$BM = 2r \sin \frac{120 - \theta}{2}$$

The maximum length tree will occur when the derivative of $AM + BM = 0$. So

$$\frac{d}{d\theta}(AM + BM) = \frac{d}{d\theta}(2r \sin \frac{\theta}{2} + 2r \sin \frac{120 - \theta}{2})$$
$$\frac{d}{d\theta}(AM + BM) = r \cos \frac{\theta}{2} - r \cos \frac{120 - \theta}{2}$$

Let $\frac{d}{d\theta}(AM + BM) = 0$. Then

$$r \cos \frac{\theta}{2} - r \cos \frac{120 - \theta}{2} = 0$$
$$r \cos \frac{120 - \theta}{2} = r \cos \frac{\theta}{2}$$
$$\cos \frac{120 - \theta}{2} = \cos \frac{\theta}{2}$$
$$\frac{120 - \theta}{2} = \frac{\theta}{2}$$
$$120 - \theta = \theta$$
$$\theta = 60^\circ$$

Then because $\angle AOB$ is 120° and θ is 60°, OM bisects $\angle AOB$. Therefore M is equidistant from A and B on the $\partial L(A, B)$, and is a point that maximizes the sum of $AC + CB$ for all C in the lune. Also by reflection, we know that the other point that maximizes this sum M^* is also equidistant from A and B on the $\partial L(A, B)$. □

5. Comparison of Degenerate Regions to Lune Region

(Setup the problem.)

5.1. **Horizontal and Vertical Case.** Given AB parallel to the x-axis, for any C in the Lune, let $L'(A,B)$ be a translate of the lune that contains C'. The maximum value of $AC+CB$ will occur when $C = M$, where M is a point that is on the $\partial(A,B)$ and equidistant from A and B, as discussed above. If b is half of the length of AB, then $AM + MB = \frac{4b}{\sqrt{3}}$.

If C is in L(A,B), the closest C' that is in the degenerate regions of AB is the same point as a translate of A (picture). If this is the case, then $AB + BC' = l$, the length of the fundamental domain. So if $\frac{4b}{\sqrt{3}} < l$, then $AC + CB$ will always be minimal for C in the lune of a given A and B.

A similar case is when AB is parallel to the y-axis. If $\frac{4b}{\sqrt{3}} < w$ the width of the fundamental domain, then $AC + CB$ will be minimal for a given A and B with C in $L(A,B)$.

5.2. **Non-Horizontal Case.** If AB is not parallel to the??? x or y-axis,??? then there is an angle, α, between AB and the positive x-axis. The midpoint of AB, D, will be located at $(b\cos\alpha, b\sin\alpha)$, and translates of D will be at $(b\cos\alpha+l, b\sin\alpha)$, $(b\cos\alpha, b\sin\alpha + w)$, and $(b\cos\alpha + l, b\sin\alpha + w)$. These are the only translates we have to consider because the minimal path network must be contained in one fundamental domain. The point B will be at $(2b\cos\alpha, 2b\sin\alpha)$.

Define the circle of rotation, P, to be the circle centered at D with radius b. The circle of rotation will circumscribe of $L(A,B)$ about the point D. (Picture)

Theorem 5.1. *Suppose A and B are points on a fundamental domain of dimension $l \times w$, in the xy-plane and let α be the angle between AB and the positive x-axis. Suppose C' is a translate of C in $L(A,B)$ in $T_{0,1}$. Then $AB+BC'$ will be less than or equal to $b + \sqrt{b^2 - 2bl\cos\alpha + l^2}$.*

Proof. By the distance formula, we know that the distance from B to D is

$$BD = \sqrt{(b\cos\alpha - l)^2 + (b\sin\alpha)^2}.$$

Then the distance from B to the edge of P' (define.. the circle of rotation) is the distance from B to the center of P' minus the radius of the circle, or

$$\sqrt{(b\cos\alpha - l)^2 + (b\sin\alpha)^2} - b.$$

Then BC' must be less than or equal to the distance from B to P' because C' must be inside P'. Then if we add AB to both sides and simplify,

$$\begin{aligned}
AB + BC' &\le 2b + \sqrt{(b\cos\alpha - l)^2 + (b\sin\alpha)^2} - b \\
AB + BC' &\le b + \sqrt{(b\cos\alpha - l)^2 + (b\sin\alpha)^2} \\
AB + BC' &\le b + \sqrt{b^2\cos^2\alpha - 2bl\cos\alpha + l^2 + b^2\sin^2\alpha} \\
AB + BC' &\le b + \sqrt{b^2 - 2bl\cos\alpha + l^2}
\end{aligned}$$

□

By a similar argument, for a translate C'' in $T_{1,0}$, $AB+BC'' \le b+\sqrt{b^2 - 2bw\sin\alpha + w^2}$.

Theorem 5.2. *If the ratio $\frac{l}{AB} > \frac{1}{2}\cos\alpha(1 + \sqrt{1 + \frac{16-8\sqrt{3}}{3\cos^2\alpha}})$ then $AC + CB < AB + BC'$ for every C in $L(A,B)$ and C', a translate of C in $T_{0,1}$.*

Proof. We know that $AB+BC' \leq b+\sqrt{b^2-2bl\cos\alpha+l^2}$, and that $AB+BC$ has maximal value $\frac{4b}{\sqrt{3}}$. So when

$$\begin{aligned}
AC+CB &< AB+BC' \\
\frac{4b}{\sqrt{3}} &< b+\sqrt{b^2-2bl\cos\alpha+l^2} \\
\frac{4b}{\sqrt{3}}-b &< \sqrt{b^2-2bl\cos\alpha+l^2} \\
(\frac{4b}{\sqrt{3}}-b)^2 &< b^2-2bl\cos\alpha+l^2 \\
(\frac{4}{\sqrt{3}}-1)^2b^2 &< b^2-2bl\cos\alpha+l^2 \\
(\frac{4\sqrt{3}-3}{3})^2b^2-b^2-l^2 &< -2bl\cos\alpha \\
(\frac{16}{3}-\frac{8}{\sqrt{3}}+1-1)b^2-l^2 &< -2bl\cos\alpha \\
\frac{(\frac{16}{3}-\frac{8}{\sqrt{3}})b^2-l^2}{-2bl} &> \cos\alpha \\
\frac{l^2+(\frac{16}{3}-\frac{8}{\sqrt{3}})b^2}{2bl} &> \cos\alpha \\
\frac{l}{2b}-(\frac{16}{3}-\frac{8}{\sqrt{3}})\frac{b}{l} &> \cos\alpha \\
\frac{l}{2b}-(\frac{8-4\sqrt{3}}{6})\frac{2b}{l} &> \cos\alpha
\end{aligned}$$

If $x=\frac{l}{2b}$, then to solve for x

$$\begin{aligned}
x-(\frac{8-4\sqrt{3}}{6})\frac{1}{x}-\cos\alpha &= 0 \\
x^2-(\frac{8-4\sqrt{3}}{6})-x\cos^2\alpha &= 0
\end{aligned}$$

Then by the quadratic formula

$$\begin{aligned}
\frac{l}{2b} &= \frac{\cos\alpha\pm\sqrt{\cos^2\alpha-4(1)(\frac{8-4\sqrt{3}}{6})}}{2} \\
\frac{l}{2b} &= \frac{\cos\alpha\pm\sqrt{\cos^2\alpha-(\frac{16-8\sqrt{3}}{3})}}{2} \\
\frac{l}{2b} &= \frac{\cos\alpha(1\pm\sqrt{1-\frac{16-8\sqrt{3}}{3\cos^2\alpha}})}{2}
\end{aligned}$$

And since only the positive values have meaning in this case (a better way to say this?)

$$\frac{l}{AB} = \frac{\cos\alpha(1+\sqrt{1-(\frac{16-8\sqrt{3}}{3\cos^2\alpha}}))}{2}$$

□

Similarly, for C'' in $T_{1,0}$, if the ratio

$$\frac{w}{AB} > \frac{1}{2}\sin\alpha(1+\sqrt{1+\frac{16-8\sqrt{3}}{3\sin^2\alpha}})$$

then $AC + CB < AB + BC''$ for every C in $L(A, B)$.

Theorem 5.3. *Given the points A, B, on the flat torus and C in $L(A, B)$, a translate of C in $T_{1,1}$ is not the closest translate of C. (A better way to say this . . . ?)*

Proof. Consider D the center of the circle of rotation in $T_{0,0}$, D' in $T_{0,1}$, and D''' in $T_{1,1}$. Assume D and D' lie on the x-axis.

If b is the radius of the circle of rotation, b cannot be larger than $\frac{1}{2}\sqrt{l^2+w^2}$, because otherwise $\overline{AB}$ does not lie in a single fundamental domain.

Consider when $b = \frac{1}{2}\sqrt{l^2+w^2}$. Let H be the circle of radius $\frac{1}{2}\sqrt{l^2+w^2}$ centered and D' and let Q a circle of the same radius centered at D'''.

Let d be a point on the circle of rotation centered at D, such that $0 \leq \angle drr' < \frac{\pi}{4}$. The point d is contained in the circle H because if you consider the line $x = \frac{1}{2}l$, then the boundary of the circle is on the left but the point d is on the right. Also the point d is not contained in the circle Q because d is below the tangent line $y = \frac{-w}{l}x + w$, so d cannot be inside Q. This implies that D' is closer to d than D'''.

Similarly, if d is on the arc of the circle so that $\frac{\pi}{4} < \angle drr' \leq \frac{\pi}{2}$, then D'' in the vertical domain will be closer than D'''. And if $\alpha = \frac{\pi}{4}$, then d is the midpoint of the diagonal of the fundamental domain, which implies that d is equidistant from the three translates of D.

Now if b be less than $\frac{1}{2}\sqrt{l^2+w^2}$ and $\frac{\pi}{4} \leq \alpha \leq \frac{\pi}{2}$. Under these conditions, the maximum value for D'' and d is at $\alpha = \frac{\pi}{4}$. What now? □

6. Comparison of Steiner Regions to Lune Region

In this section we will be comparing the length of the maximal lune region tree, the union of $\overline{AM}$ and $\overline{MB}$, with trees in the Steiner region constructed with a translate of C, a point in the lune region. For the length of the Steiner tree we will use the Simpson line, the line from E to C'. Again we will denote b as half of the length of AB. Given this, the maximum length tree in the lune is $\frac{4b}{\sqrt{3}}$. Denote lune focus above $\overline{AB}$ to be O and corresponding translate to be O'.

Theorem 6.1. *Suppose, A and B are points in a fundamental domain of a torus of dimension $l \times w$, $\overline{AB}$ makes an angle α with the positive x-axis. A new set of axes* *are setup with the center of the lune centered at the origin, then $\overline{DD'}$, the segment from the center of the lune to the center of the translated lune makes an angle β with the positive x'-axis. (Note that β is also the angle between a horizontal line through O and $\overline{OO'}$.) Suppose E is the E-point for A and B below $\overline{AB}$, then for any C' in a translate of $L(A,B)$,*

$$EC' \leq \sqrt{l^2 + \frac{8bl}{\sqrt{3}}\sin\beta + \frac{16b^2}{3}} - \frac{2b}{\sqrt{3}}.$$

Proof. By the distance formula, we know that the distance from E to M' is

$$\sqrt{(l\cos\beta)^2 + (l\sin\beta + \frac{b}{\sqrt{3}} + \sqrt{3}b)^2}.$$

Then the distance from C' to M' is the radius of the circle, or $\frac{2b}{\sqrt{3}}$. Therefore EC' must be less than or equal to $EM' - C'M'$, when we simplify we get $EC' \leq \sqrt{l^2 + \frac{8bl}{\sqrt{3}}\sin\beta + \frac{16b^2}{3}} - \frac{2b}{\sqrt{3}}$

□

By a similar argument, for a translate in the vertical domain

$$EC'' \leq \sqrt{w^2 + \frac{8bw}{\sqrt{3}}\sin(90-\beta) + \frac{16b^2}{3}} - \frac{2b}{\sqrt{3}}.$$

Lemma 6.2. $r^2 - \frac{2\sqrt{3}}{5}r\sin\beta - 3/20 > 0$ *if and only if* $EC' > AC + CB$ *for positive* r.

Proof. We know that the maximum length of $AC + CB$ inside the $L(A,B)$ is $\frac{4b}{\sqrt{3}}$. We want that length to be less than EC'.

$$\frac{4b}{\sqrt{3}} < \sqrt{l^2 + \frac{8bl}{\sqrt{3}}\sin\beta + \frac{16b^2}{3}} - \frac{2b}{\sqrt{3}}$$
$$(\frac{6b}{\sqrt{3}})^2 < l^2 + \frac{8bl}{\sqrt{3}}\sin\beta + \frac{16b^2}{3} - \frac{2b}{\sqrt{3}}$$
$$\frac{36b^2}{3} < l^2 + \frac{8bl}{\sqrt{3}}\sin\beta + \frac{16b^2}{3}$$
$$\frac{20b^2}{3} - \frac{8bl}{\sqrt{3}}\sin\beta - l^2 > 0$$

If we divide this by $\frac{3l^2}{20}$ and note that $r = b/l$, then we get

$$r^2 - \frac{2\sqrt{3}}{5}r\sin\beta - 3/20 > 0.$$

□

Lemma 6.3. *If C' is in the Steiner region, then cos*$(\beta + 30°) \leq 2r/\sqrt{3}$

Proof. We want to find what angle β is when M' is on the boundary between the Steiner region and the degenerate region. Note that β is also the angle between the line parallel to the x-axis through point $M$and $\overline{MM'}$. Denote K to be the intersection of the line parallel to the x-axis through point M and $\overline{BM'}$. We draw $\triangle BMM'$. The length of MM' is l, and the length of MB is $\frac{2b}{\sqrt{3}}$. Denote $\angle BMM'$ as δ. We know that

$$\cos\delta = \frac{2b/\sqrt{3}}{l} = \frac{2b}{\sqrt{3}l}.$$

It is easy to find that $\angle KMB$ is 30°. In order for point M' to be in the Steiner region

$$\beta \geq \arccos\frac{2r}{\sqrt{3}} - 30°.$$

From this we see that

$$\cos(\beta + 30°) \leq \frac{2r}{\sqrt{3}}.$$

□

Theorem 6.4. *Let $r = \frac{b}{l}$, if $r \leq \frac{\sqrt{63+14\sqrt{15}}}{14}$ then $AC + CB < EC'$ for every C in $L(A, B)$ and C', a translate of C in the horizontal domain.*

Proof. From before, we know that

$$r^2 - \frac{2\sqrt{3}}{5}r\sin\beta - \frac{3}{20} > 0.$$

We can us the quadratic formula to solve for r.

$$r > \frac{\frac{2\sqrt{3}}{5}\sin\beta \pm \sqrt{\frac{12}{25}(\sin\beta)^2 + 4\frac{3}{20}}}{2}$$

We do not want r to be negative, so

$$r > \frac{\sqrt{3}}{5}\sin\beta + \sqrt{\frac{3}{25}(\sin\beta)^2 + \frac{3}{20}}.$$

Now we want to solve $\cos(\beta + 30°) \leq \frac{2r}{\sqrt{3}}$ for $\sin\beta$. By using the cosine sum formula we get,

$$\cos\beta\cos 30^\circ - \sin\beta\sin 30^\circ \leq \frac{2r}{\sqrt{3}}$$
$$\frac{\sqrt{3}}{2}\cos\beta - \frac{1}{2}\sin\beta \leq \frac{2r}{\sqrt{3}}$$
$$\frac{\sqrt{3}}{2}\sqrt{1-(\sin\beta)^2} - \frac{1}{2}\sin\beta \leq \frac{2r}{\sqrt{3}}$$
$$(\frac{\sqrt{3}}{2}\sqrt{1-(\sin\beta)^2})^2 \leq (\frac{1}{2}\sin\beta + \frac{2r}{\sqrt{3}})^2$$
$$\frac{3}{4}(1-(\sin\beta)^2) \leq \frac{1}{4}(\sin\beta)^2 + \frac{2r}{\sqrt{3}}\sin\beta + \frac{4r^2}{3}$$
$$(\sin\beta)^2 + \frac{2r}{\sqrt{3}}\sin\beta + \frac{16r^2-9}{12} \geq 0$$

We can use the quadratic formula to solve for $\sin\beta$.

$$\sin\beta \leq \frac{\frac{-2r}{\sqrt{3}} \pm \sqrt{\frac{4r^2}{3} - 4\frac{16r^2-9}{12}}}{2}$$

We do not want $\sin\beta$ to be negative, so

$$\sin\beta \leq \frac{-r}{\sqrt{3}} + \sqrt{\frac{3-4r^2}{4}}.$$

When we put this in for $\sin\beta$ in our r equation. We find that

$$r \leq \frac{\sqrt{63-14\sqrt{15}}}{14} \approx .77.$$

Therefore if

$$r \leq \frac{\sqrt{63-14\sqrt{15}}}{14}$$

, then $EC' > AC + CB$.

A similar result can be found for the points in the $T_{0,1}$ fundamental domain, exchanging w for l and letting β be the angle between the negative x-axis and the line from O to O''. □

7. Summary

We have found that given a torus covered by a fundamental domain of , and given two points A and B separated by a distance of . If a point lies in the $L(A, B)$ then the minimal path network is the union of segments connecting A to C and C to B.

8. Conclusion

These results are not tight results, but they are a start. We have experimented with this problem in Geosketchpad and in Maple, and it appears that these results are true for greater values of AB, almost for any A and B contained in one fundamental domain. In the future we hope to find tighter results, use this to completely solve the three point problem on the torus, and to extend the three point problem to n points on the flat torus.