STEINER TREE CONSTRUCTIONS IN HYPERBOLIC SPACE

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Abstract. Methods for the construction of Steiner minimal trees for n fixed points in the hyperbolic plane are developed. A brief explanation of Melzak's solution for the Euclidean problem is given. One method of construction in the hyperbolic plane identifies Steiner points as the solution to a system of equations. This method is derived partially by analogous reasoning to Melzak. A second method uses numerical approximation to identify the Steiner points. The properties of Steiner minimal trees in the hyperbolic plane are investigated.

Contents

1. Introduction

What is the least length necessary to construct a path joining an arbitrary number of fixed points? This question in geometric optimization has come to be known as the Steiner problem, and there have been numerous articles written on the topic. Melzak essentially settled the question for the Euclidean plane by describing an algorithm that will, in a finite number of steps, give the exact minimal path connecting an arbitrary number of points. Several well-known survey publications are [1], [2] and [3]. The topic is now researched by many computer scientists, as the problem is known to be NP-hard.

While the Steiner problem in Euclidean space has been studied to a great extent, very few results have been found in the hyperbolic plane. In this paper we present algorithms for the construction of Steiner minimal trees in hyperbolic space.

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Specifically, methods are presented for the upper-half plane and Poincare disk models of the hyperbolic plane. The methods developed are applicable to any general topological space—spheres, tori, and even higher dimensions of space.

After developing the construction algorithms, we briefly analyze the characteristics of Steiner trees in hyperbolic space. A new numerical method of approximating the Steiner points in any topological setting is given at the end of the paper.

2. Preliminaries

The terms *points* and *vertices* will be used interchangeably, but we will typically refer to points when discussing geometry and vertices when referring to topological/graph theoretic properties. We use $\mathcal{P}$ to denote a set of points $P_1, \ldots, P_n$.

We borrow several terms from graph theory:

Definition 2.1. If there is an edge between two vertices, they are said to be *adjacent*. The *degree* of a vertex is the number of adjacent vertices.

Definition 2.2. A *tree* is a network (graph) with no cycles; that is, given two vertices in the network there is only one path between the two. A network is said to *span* a set of vertices $\mathcal{P}$ if for all $P_i, P_j \in \mathcal{P}, i \neq j$, there is a path between P_i and P_j.

If there is no restriction against the introduction of new vertices, the length required to span $\mathcal{P}$ can often be reduced (See Figure 1).

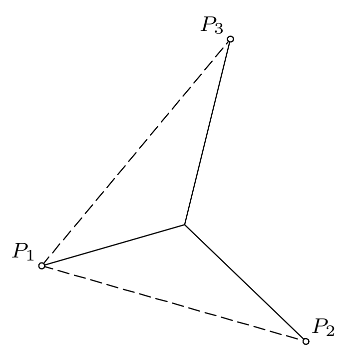

Figure 1. Length reduced by introduction of intermediate point.

Intermediate points that are introduced to shorten the length are called *Steiner points*. The points of $\mathcal{P}$ are referred to as *fixed points* to distinguish them from the Steiner points, and since their location is constant for all spanning networks.

Definition 2.3. Let $\mathcal{P}$ be a set of points in the plane and let G be the graph of a tree that spans $\mathcal{P}$. If each vertex of $\mathcal{P}$ has degree 1 and all other vertices of G have degree 3 or less, then G is called a Steiner topology.

Clearly length minimizers are trees since the network could lose any edge of a cycle and still span the set of points.

Definition 2.4. Let G be a Steiner topology. If a tree has topology G and its length cannot be decreased by any small fluctuation of the vertices, then the tree is called a *Steiner tree* (ST). A tree with shortest possible length for a given topology is called the *relatively minimal tree* with topology G.

Definition 2.5. Let $\mathcal{P}$ be a collection of vertices. The shortest ST that spans $\mathcal{P}$ is called the *Steiner minimal tree* (SMT).

Note that a relatively minimal tree need not exist for every topology, but that a SMT is necessarily a relatively minimal tree for its topology.

Observation 2.6. A Steiner tree has at most $n-2$ Steiner points, and no two edges of an ST can meet at an angle with measure less than $120°$.

See [2] for the proofs of these observations.

Finding the SMT on a set of fixed points in the Euclidean plane is commonly referred to as the Euclidean Steiner problem (ESP). In this paper, the same problem in the hyperbolic plane will be referred to as the hyperbolic Steiner problem (HSP). To model the hyperbolic plane, we use the upper-half plane model.

Definition 2.7. The *upper-half plane* consists of the region $y > 0$, while the line $y = 0$ represents points at infinity. To model hyperbolic space, the metric that is used has arclength element $\sqrt{dx + \frac{1}{dy}}$. A *geodesic* is the shortest length between two points. Using this metric, a geodesic is a Euclidean arc with center along the line $y = 0$.

The center of arc from P_1 to P_2 can be found by finding the intersection of the perpendicular bisector of $\overline{P_1P_2}$ with the x-axis.

3. Melzak's Algorithm

3.1. **Overview of the Algorithm.** Melzak [4] found a purely geometric algorithm for the construction of an SMT in the Euclidean plane. As our methods for the hyperbolic plane follow similar reasoning, we present an overview of Melzak's findings. Specifically, we give the algorithm in the simplified case of finding the minimal tree on three fixed points.

Let P_1, P_2, and P_3 be three fixed points in the Euclidean plane. Select two of the points, say P_1 and P_2, and construct an equilateral triangle that has $\overline{P_1P_2}$ as one of its legs. There are two possible orientations for this triangle, and so the third vertex should be chosen so that it falls on the opposite side of $\overline{P_1P_2}$ as P_3. This third vertex is called the equilateral-point, hereafter referred to as the *E-point*, and we denote it as E (see Figure 2).

Construct the circle the circumscribes $\triangle P_1P_2E$. Call this circle C. The center O is easily located by finding the intersection of the perpendicular bisectors of the sides of $\triangle P_1P_2E$. Consider the arc $\widehat{P_1P_2}$ with center O. Since $\overline{P_1P_2}$ is a side of an equilateral triangle, it subtends $120°$. For any point S on $\widehat{P_1P_2}$, $m\angle P_1SP_2 = 120°$. Therefore, $\widehat{P_1P_2}$ represents the locus of points that form a $120°$ angle with P_1 and P_2, and the Steiner point must lie along this arc (see Figure 3).

Construct the line segment $\overline{EP_3}$ and call S the point of intersection of this segment with $\widehat{P_1P_2}$. As discussed above, $m\angle P_1SP_2 = 120°$. Since $\angle P_1SE$ and $\angle P_2SE$ subtend arcs of the same length, they have the same measure. Thus, $\angle P_1SE = 60° = \angle P_2SE$ and $\angle P_1SP_3 = 120° = P_2SP_3$ (see Figure 4).

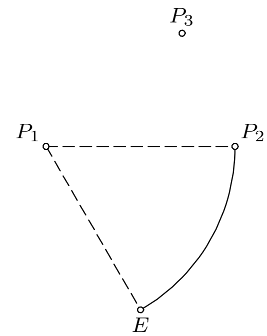

FIGURE 2. Construction of the E-point.

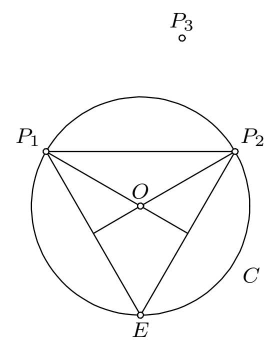

FIGURE 3. Circumscribing $\triangle P_1P_2E$.

The point S has $120°$ angles surrounding it, and therefore must be the Steiner point. It is convenient that the segment $\overline{EP_3}$ (called the *Simpson line*) has the same length as the Steiner tree; that is, $EP_3 = P_1S + P_2S + P_3S$ (see Figure 5).

3.2. **Challenges in Extending Melzak's Algorithm.** In the Euclidean plane, it is possible to find the locus of points that form a $120°$ angle with two fixed points by a ruler and compass construction. In the most common models of the hyperbolic plane—the Poincare disk and the upper-half plane—there is not an analogous process available. As will be seen later, the locus is not an elementary curve.

The main difficulty in adapting the construction detailed above to the hyperbolic plane is due to the lack of an E-point. In the Euclidean plane, the Steiner point can be found be drawing a segment from the E-point to the third fixed point. Since this cannot be done to find a Steiner tree in the hyperbolic plane, we show how the Steiner point could be found in the Euclidean case by repeating the first part of the construction.

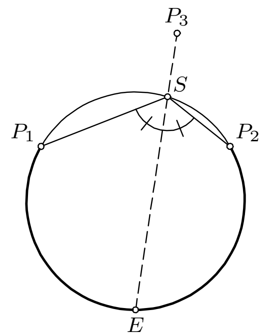

FIGURE 4. Properties of $\overline{EP_3}$.

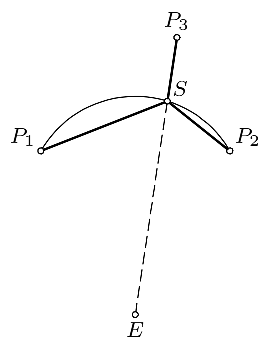

FIGURE 5. The resulting ST.

3.3. **Adapting the Algorithm.** After constructing C, the circle which circumscribes $\triangle P_1P_2E$, construct the equilateral triangle that has $\overline{P_2P_3}$ as one of its legs. Again, choose the third vertex of the triangle to lie on the opposite side of $\overline{P_2P_3}$ as P_1. Circumscribe this new triangle in the same way as $\triangle P_1P_2E$ (see Figure 6).

The arc $\widehat{P_2P_3}$ represents the locus of points that form a 120° angle with P_2 and P_3. Therefore, the intersection of $\widehat{P_1P_2}$ and $\widehat{P_2P_3}$ forms 120° angles with P_1, P_2, and P_3. Since the Steiner point is unique for three fixed points, this is the point S that was previously found by use of the E-point.

4. Steiner Tree Constructions in the Upper-half Plane

Suppose that a point S forms a 120° angle with P_1 and P_2 (see Figure 7). Let C_1 be the center of $\widehat{P_1S}$ and C_2 be the center of $\widehat{P_2S}$. (We refer to these arcs since we are using Euclidean geometry for the constructions in upper half plane model; in reality the arcs are geodesics in the hyperbolic plane.) Since the radius $\overline{C_1S}$ is perpendicular to the tangent of $\widehat{P_1S}$ at S, and the same is true of the radius $\overline{C_2S}$

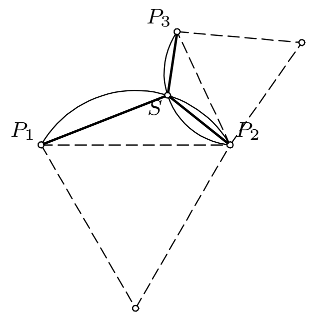

FIGURE 6. Alternative construction of the ST.

with the tangent of $\widehat{P_2S}$, we can find the measure of $\angle C_1SC_2$. Combining this with the fact that radii of an arc have the same length, we have the following three restraints:

$$m\angle C_1SC_2 = 60^\circ \tag{1}$$
$$\overline{C_1S} = \overline{C_1P_1} \tag{2}$$
$$\overline{C_2S} = \overline{C_2P_2}. \tag{3}$$

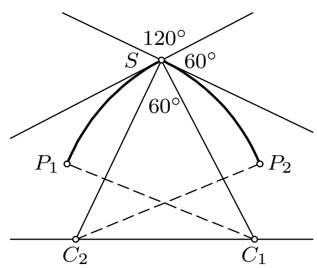

FIGURE 7. Restraints on the definition of a Steiner point.

Definition 4.1. The collection of all points S that satisfy the condition $m\angle P_1SP_2 = 120^\circ$ is called the locus of P_1 and P_2. We denote the locus of P_1 and P_2 by $L(P_1, P_2)$.

Let $P_1 = (x_1, y_1)$, $P_2 = (x_2, y_2)$, $C_1 = (c_1, 0)$ and $C_2 = (c_2, 0)$. We find an equation for $L(P_1, P_2)$ that satisfies the three conditions given above.

From (1) we have

$$\frac{\overrightarrow{SC_1} \cdot \overrightarrow{SC_2}}{\left\|\overrightarrow{SC_1}\right\| \left\|\overrightarrow{SC_2}\right\|} = \cos 60^\circ = \frac{1}{2}.$$

Using equations (2) and (3) and substituting the coordinates,

$$\frac{\overrightarrow{SC_1}\cdot\overrightarrow{SC_2}}{\left\|\overrightarrow{SC_1}\right\|\left\|\overrightarrow{SC_2}\right\|}=\frac{\overrightarrow{SC_1}\cdot\overrightarrow{SC_2}}{\left\|\overrightarrow{C_1P_1}\right\|\left\|\overrightarrow{C_2P_2}\right\|}$$
$$=\frac{< c_1-x,-y>\cdot< c_2-x,-y>}{\sqrt{(c_1-x_1)^2+(-y_1)^2}\sqrt{(c_2-x_2)^2+(-y_2)^2}}$$
$$=\frac{(c_1-x)(c_2-x)+y^2}{\sqrt{\left((x_1-c_1)^2+{y_1}^2\right)\left((x_2-c_2)^2+{y_2}^2\right)}}.$$

Therefore,

$$y^2=\frac{1}{2}\sqrt{\left((x_1-c_1)^2+{y_1}^2\right)\left((x_2-c_2)^2+{y_2}^2\right)}-(c_1-x)(c_2-x)\,. \tag{4}$$

Since c_1 and c_2 are uniquely determined by P_1, P_2, and S, we seek substitutions to reduce the number of parameters.

$(x_1-c_1)^2+{y_1}^2=(x-c_1)^2+y^2{x_1}^2-2x_1c_1+{c_1}^2+{y_2}^2=x^2-2xc_1+{c_1}^2+y^2$

It follows that

$$c_1=\frac{1}{2}\cdot\frac{x^2-{x_1}^2+y^2-{y_1}^2}{x-x_1}$$

and similarly,

$$c_2=\frac{1}{2}\cdot\frac{x^2-{x_2}^2+y^2-{y_2}^2}{x-x_2}.$$

Substituting back into (4), we find

$$y^2=\frac{1}{2}\sqrt{\left(\left(x_1-\frac{1}{2}\cdot\frac{x^2-{x_1}^2+y^2-{y_1}^2}{x-x_1}\right)^2+{y_1}^2\right)\left(\left(x_2-\frac{1}{2}\cdot\frac{x^2-{x_2}^2+y^2-{y_2}^2}{x-x_2}\right)^2+{y_2}^2\right)}$$
$$-\left(\frac{1}{2}\cdot\frac{x^2-{x_1}^2+y^2-{y_1}^2}{x-x_1}-x\right)\left(\frac{1}{2}\cdot\frac{x^2-{x_2}^2+y^2-{y_2}^2}{x-x_2}-x\right).$$

This equation gives an implicit formula for part of $L(P_1,P_2)$. However, due to distortions caused by the upper-half plane model, it is also necessary to consider points S where

$$\frac{\overrightarrow{SC_1}\cdot\overrightarrow{SC_2}}{\left\|\overrightarrow{SC_1}\right\|\left\|\overrightarrow{SC_2}\right\|}=\cos 120^\circ=-\frac{1}{2}.$$

An illustration of why this is necessary is given in Figure 8 where $m\angle P_1SP_2=m\angle C_1SC_2=120^\circ$ (see Figure 8).

We define $H(S,P_1,P_2)$ to be

$$\pm\frac{1}{2}\sqrt{\left(\left(x_1-\frac{1}{2}\cdot\frac{x^2-{x_1}^2+y^2-{y_1}^2}{x-x_1}\right)^2+{y_1}^2\right)\left(\left(x_2-\frac{1}{2}\cdot\frac{x^2-{x_2}^2+y^2-{y_2}^2}{x-x_2}\right)^2+{y_2}^2\right)}$$
$$-\left(\frac{1}{2}\cdot\frac{x^2-{x_1}^2+y^2-{y_1}^2}{x-x_1}-x\right)\left(\frac{1}{2}\cdot\frac{x^2-{x_2}^2+y^2-{y_2}^2}{x-x_2}-x\right)-y^2.$$

so that $S\in L(P_1,P_2)$ only if $H(S,P_1,P_2)=0$ (see Figure 9).

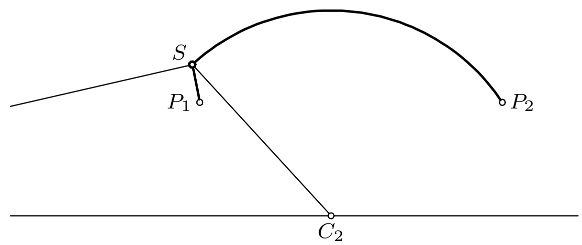

FIGURE 8. An 120° angle where $m\angle C_1SC_2 = 120°$.

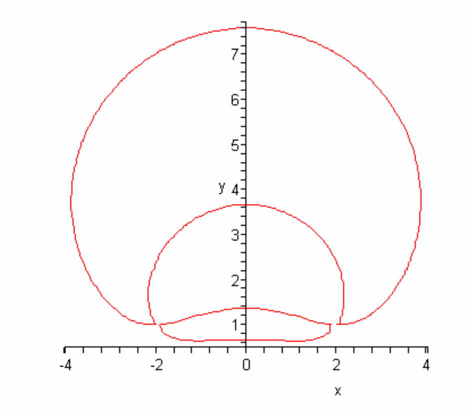

FIGURE 9. Plot of $H(S, P_1, P_2) = 0$ where $P_1 = (-2, 1)$ and $P_2 = (2, 1)$.

Unfortunately, $H(S, P_1, P_2)$ includes points that form 120° and 60° angles with P_1 and P_2. We therefore restrict our attention to the part of the plot that gives $L(P_1, P_2)$ (see Figure 10).

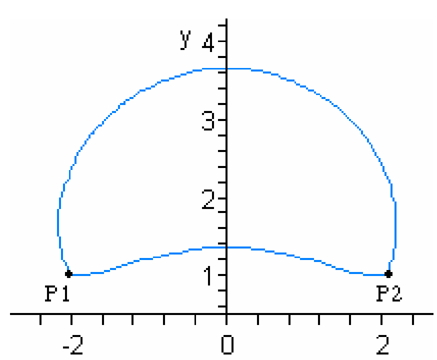

FIGURE 10. $L(P_1, P_2)$ where $P_1 = (-2, 1)$ and $P_2 = (2, 1)$.

As defined previously, $L(P_1, P_2)$ represents all points S that form a 120° with P_1 and P_2. Therefore, a point that satisfies $H(S, P_1, P_2) = 0$ and $H(S, P_2, P_3) = 0$ must lie on the intersection of $L(P_1, P_2)$ and $L(P_2, P_3)$, and is therefore a Steiner point (see Figure 11).

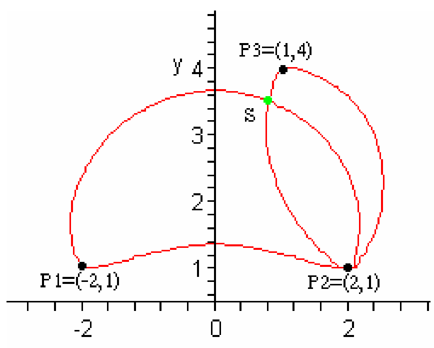

Figure 11. Intersection of $L(P_1, P_2)$ and $L(P_2, P_3)$.

Once the Steiner point is located, it is a simple geometric construction to draw the geodesics of the Steiner tree (see Figure 12).

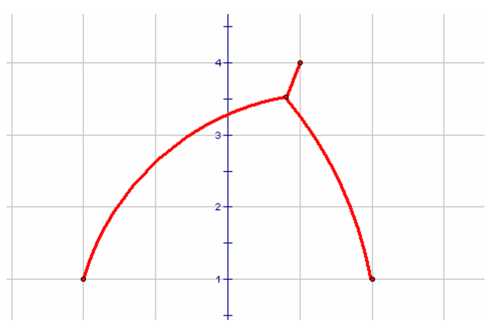

Figure 12. ST constructed on three points in the hyperbolic plane.

This method can be extended to suit an arbitrary number of fixed points. A full Steiner tree on n fixed points will have $n - 2$ Steiner points. Each of these must form 120° angles with the three vertices adjacent to it. This can be expressed mathematically by

$$\begin{aligned} H(S_1, N_{1_1}, N_{2_1}) &= 0 \\ H(S_1, N_{2_1}, N_{3_1}) &= 0 \\ &\vdots \\ H(S_i, N_{1_i}, N_{2_i}) &= 0 \\ H(S_i, N_{2_i}, N_{3_i}) &= 0 \\ &\vdots \\ H(S_{(n-2)}, N_{1_{(n-2)}}, N_{2_{(n-2)}} &= 0 \\ H(S_{(n-2)}, N_{2_{(n-2)}}, N_{3_{(n-2)}} &= 0 \end{aligned}$$

where N_{i_1}, N_{i_2}, and N_{i_3} are the vertices adjacent to S_i. Thus, the solution to a Steiner tree on n vertices requires solving a system of $2(n-2)$ simultaneous equations.

5. A Numerical Algorithm

The following algorithm is proved by several arguments using the relative measures of angles. At each step it should be noted that the validity of the argument does not depend on the parallel postulate; thus, while the pictures show an application to the ESP, the algorithm also holds for hyperbolic and spherical space.

Definition 5.1. Let A, B, and C be three fixed points. Define $M(A, B, C)$ to be the location of the Steiner point for the SMT on A, B, and C.

Note that M is well-defined since there is a unique intermediate point for the SMT.

Suppose a_1 is a point on $L(P_1, P_2)$. Let $b_1 = M(a_1, P_1, P_2)$. We recursively define $a_{i+1} = M(P_1, P_2, b_i)$ and $b_{i+1} = M(P_3, P_4, a_{i+1})$ (see Figure 13).

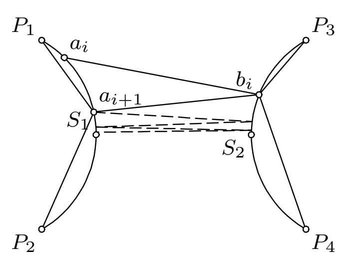

Figure 13. Definition of $\{a_n\}$ and $\{b_n\}$.

Since for any four fixed points there is a unique tree that has 120° angles at the Steiner points, we can show that these sequences of points $\{a_n\}$ and $\{b_n\}$ converge to S_1 and S_2, respectively.

Lemma 5.2. *All terms of a_n and b_n lie on the same side of $\overline{S_1 S_2}$.*

Proof: If b_i were to lie on the opposite side of a_i, then $m\angle a_i S_2 P_4 < m\angle a_i b_i P_4$. Also, $m\angle S_1 S_2 P_4 < m\angle a_i S_2 P_4$. But by hypothesis $m\angle a_i b_i P_4$ and $m\angle S_1 S_2 P_4$ are both 120°, so we have $120° = m\angle S_1 S_2 P_4 < m\angle a_i S_2 P_4 < m\angle a_i b_i P_4 = 120°$, a contradiction (see Figures 14 and (see Figure 15)).

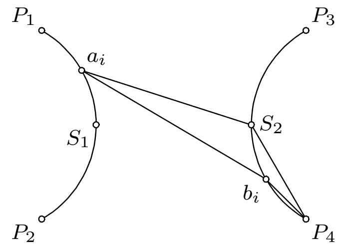

FIGURE 14. $m\angle a_i S_2 P_4 < m\angle a_i b_i P_4$.

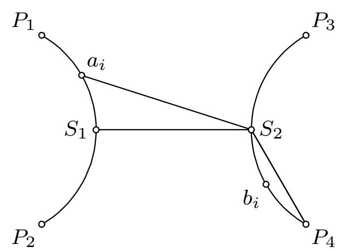

FIGURE 15. $m\angle S_1 S_2 P_4 < m\angle a_i S_2 P_4$.

Lemma 5.3. *The point b_{i+1} lies on the same side of $\overline{a_i b_i}$ as a_{i+1}.*

Proof: By the previous lemma, if a_{i+1} is on the same side of $\overline{a_i b_i}$ as S_1, then b_{i+1} is as well. Since $m\angle a_i b_i P_4 = 120°$, $m\angle a_{i+1} b_i P_4 < 120°$. Were b_{i+1} to lie on the other side of $\overline{a_i b_i}$, then the angle would decrease again, so that $m\angle a_{i+1} b_{i+1} P_4 < 120°$. A symmetric argument proves the case that a_{i+1} is on the opposite side of $\overline{a_i b_i}$ as S_1 (see Figure 16).

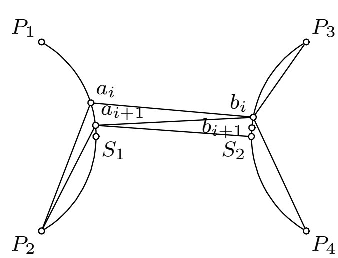

FIGURE 16. The point b_{i+1} lies on the same side of $\overline{a_i b_i}$ as a_{i+1}.

Theorem 5.4. *The sequences $\{a_n\}$ and $\{b_n\}$ converge to S_1 and S_2, respectively.*

Proof: From the lemmas, $\{a_n\}$ and $\{b_n\}$ are monotonic—they move in only one direction along $L(P_1, P_2)$ and $L(P_3, P_4)$. On one side they are bounded by P_1 and P_2, and on the other they are bounded by S_1 and S_2. Since the sequences are monotonic and bounded, they must converge to some points A and B. However, by definition of the sequences $A = M(P_1, P_2, B)$ and $B = M(P_3, P_4, A)$. This requires that the angles around A and B are all $120°$, and by the uniqueness of the Steiner tree, $A = S_1$ and $B = S_2$.

This algorithm can easily be extended to find the ST on more than three points by a straightforward use of recursion. The benefit of doing so is that the problem is reduced to a repeated solution of the three point problem. In some instances this may take much less computational time than solving a system of $2(n-2)$ simultaneous quartic equations.

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