THE STEINER PROBLEM ON NARROW AND WIDE CONES

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ABSTRACT. Given a surface and n fixed points on the surface, the Steiner problem asks to find the minimal length path network in the surface connecting the n fixed points. The solution to a Steiner problem is called a Steiner minimal tree. The Steiner problem in the plane has been well studied, but until recently few results have been found for non-planar surfaces. In this paper we examine the Steiner problem on both narrow and wide cones. We prove several important properties of Steiner minimal trees on cones and present an algorithm solving the three point problem. We discuss how the algorithm for the 3 point problem can be generalized to an algorithm for the n point problem. These results are preliminary to solving the Steiner problem on piecewise linear or polyhedral surfaces, since any piecewise linear surface may be viewed as a locally finite union of overlapping cones.

1. INTRODUCTION

The Steiner problem on a surface is to find the minimal length path network on the surface connecting a specified set of points. An illustrative application of the Steiner problem is finding the cost minimizing design of telephone lines or other wiring systems. This has prompted extensive research into developing efficient algorithms for solving Steiner problems in the plane. Until recently however, few results have been obtained for non-planar surfaces, which limits areas of application. For example, many wiring projects need to account for mountainous terrain or changes in elevation.

Our results are an important step in developing an algorithm for solving Steiner problems on arbitrary piecewise-linear (or polyhedral) surfaces. Any piecewise linear surface may be viewed as a locally finite union of overlapping cones. This paper presents an algorithm for solving Steiner problem on narrow and wide cones.

2. Preliminaries

A *narrow cone* may be constructed by cutting a sector out of the plane and gluing together the edges of the remaining sector. A *wide cone* may be constructed by cutting along a ray and inserting a sector into the plane in the natural way.

¹⁹⁹¹ Mathematics Subject Classification. Primary: 05C05 Secondary 51M15.

Key words and phrases. Steiner Problem, g-minimal tree, length minimization, cone, piecewise-linear surface.

A Steiner problem begins with a specified set of points a_1, a_2, \ldots, a_n on a surface. These points are called *terminals*. It is desired to find the least length path network connecting the terminals. The desired solution must be a tree, since deleting an edge of any cycle would shorten the path network. When additional points s_1, s_2, \ldots, s_k are introduced as vertices in a tree, these additional points are called *Steiner points*. Terminals that are connected to the same Steiner point, each by a single edge of the tree, are called *siblings*. The *degree* of a point is the number of edges in a tree that meet at that point.

The Steiner minimal tree for n points a_1, a_2, \ldots, a_n on a given surface is the shortest tree on that surface connecting a_1, a_2, \ldots, a_n . A locally minimal tree is a tree whose length cannot be shortened by small perturbations of its Steiner points. Weng [6] showed that for a minimal tree on curved surfaces, edges meeting at any point of the tree must form angles of measure at least 120°, and have degree no more than 3. In particular Steiner points have degree 3 and angles formed at Steiner points have measure equal to 120°. A Steiner tree is a tree that satisfies these angle conditions. Since the Steiner minimal tree must be a minimal tree, it will always be a Steiner tree. In the plane, if a tree is a Steiner tree, this implies that it is also a minimal tree. However this is not true in general on curved surfaces, where it is possible for a Steiner point that satisfies the angle condition to be a maximal or saddle point with respect to the length of the tree. For example, let three equidistant points on the equator of a sphere be the set of terminals. A Steiner tree with its Steiner point at a pole is a maximal tree.

From a counting argument, the maximum number of Steiner points in a Steiner tree with n terminals is n - 2. A Steiner tree for n points is called *full* if every terminal has degree 1. A full Steiner tree will have the maximum n - 2 Steiner points distinct from each terminal. Gilbert and Pollack define a Steiner tree as *degenerate* if it has fewer than n - 2 Steiner points [2]. However, for simplicity of argument in this paper we assert that all Steiner trees have n - 2 Steiner points, and a degenerate Steiner tree is one in which one or more Steiner points coincide with a terminal. Thus, in a degenerate tree, the angle formed by two edges meeting at a Steiner point has measure greater than or equal to 120° .

We define a g-minimal tree to be a minimal tree where every edge of the tree is a minimal geodesic. Note that a minimal tree need not be a g-minimal tree if there exists more than one geodesic connecting two points on a surface. The Steiner minimal tree will always be a g-minimal tree.

For a given set of terminals, two trees connecting them will be said to be in the same *Steiner topology class* provided there is an isotopy from each one to the other that fixes the terminal points (hence each tree can be pushed onto the other in the surface without moving the terminals). A tree for n points a_1, a_2, \ldots, a_n is called *relatively minimum* if it is the shortest tree connecting a_1, a_2, \ldots, a_n of all trees with the same Steiner topology. Figure 1 shows two relatively minimum trees with different Steiner topologies for points A_1, A_2, A_3 , and A_4 . In Figure 1 A_1 and A_2 are siblings on the left, while on the right A_1 and A_3 are siblings.



FIGURE 1. Two combinatorial structures for four points



FIGURE 2. Three point Steiner problem in the plane

We will now describe a classical algorithm for constructing the minimal tree for three points, A, B, and C, in the plane. Suppose there exists an interior angle of $\triangle ABC$ with angle measure greater than or equal to 120°. Without loss of generality suppose $m \angle ABC \geq 120^{\circ}$. This is the degenerate case, and the minimal tree is $\overline{AB \cup BC}$ (or $\overline{AS \cup BS \cup CS}$ where S = B). If instead all interior angles of $\triangle ABC$ have measure less than 120° then the Steiner point is distinct from A, B, and C and is contained in the interior of $\triangle ABC$. The Steiner point can be found using the following construction on one of $\triangle ABC$'s edges: We use \overline{BC} . Construct the equilateral triangle $\triangle CBE$ with the point E opposite the point A with respect to CB. (For the purposes of this paper, when we refer to the '*E*-point' we are referring to the point E obtained in this construction.) Construct a circle D that circumscribes $\triangle CBE$, and then construct the line segment EA. This line segment is called the Simpson line. The intersection of the Simpson line and the circle is the Steiner point S, and the minimal tree is $AS \cup BS \cup CS$. It has also been shown by Coxeter that for a non-degenerate tree, the length of the Simpson line is equal to the total length of the minimal tree [2].

3. Properties of G-Minimal Trees on Narrow and Wide Cones

In this section we will develop the theoretical basis for our analysis of trees on cones. Following are several results that will allow us to isolate properties of g-minimal trees and of the Steiner minimal tree that apply to both narrow and wide cones.

Propositon 3.1. The Steiner minimal tree for a given set of n-points on a cone exists.

Proof. The proof is standard and follows from an application of the extreme value theorem for continuous functions over compact sets. \Box

While the above proposition applies to any set of n points on a cone, this section and the two following give results for three points on a given cone. The extension to n points is found in Section 6.

For the three point problem we will call the three terminals a, b, and c, andthe vertex of the cone v. Because of the possible confusion of angles at the vertex, we will define $\angle^* avb$ as follows: Suppose a, b, and c are distinct from the vertex v and none of \overrightarrow{va} , \overrightarrow{vb} , or \overrightarrow{vc} are the same. Let \angle^*avb be the angle whose interior is traversed as \overrightarrow{va} rotates counterclockwise to \overrightarrow{vb} , looking down along the axis of the cone. In the same way, $sector \angle^* avb$ will be the sector of the cone traversed by \overrightarrow{va} as it rotates counterclockwise to \overrightarrow{vb} . Throughout this paper, unless otherwise specified, we will assume that the three such points are labeled such that sector $\angle^* avb$ does not include c in its interior. (This last condition is really a statement about the orientation of a, b, and c.) Suppose instead that a, b, c. b, and c are distinct for the vertex but at least two of \overrightarrow{va} , \overrightarrow{vb} , and \overrightarrow{vc} coincide. We define the angles the same counter clockwise orientation applies as before. For example if $\overrightarrow{va} = v\dot{b}$, then $m\angle^*avb = 0$ and $sector\angle^*avb = \overrightarrow{va}$, while $sector\angle^*bva$ is the entire cone. In either case we will refer to a, b, and c as oriented in this triple. If one of a, b, and c coincides with the vertex v, then the angles involving that point are undefined. We will address this case separately in our proofs.

The cone angle will be defined as the unique number $\psi = m \angle xvy + m \angle yvx$ where x and y are points on the cone distinct from the vertex v. For narrow cones $\psi < 360^{\circ}$, and for wide cones $\psi > 360^{\circ}$. In the case that $\psi = 360^{\circ}$ we have a flat cone that is equivalent to the Euclidean plane.

In several of the proofs that follow we will use a projection map, rotational projection, which we define now: Let points x and y on a cone be given. We will define rotational projection as a projection of all points in $sector \angle^* xvy$ onto \overrightarrow{vx} (or alternatively \overrightarrow{vy}). For $p \in sector \angle^* xvy$, its projection $p' \in \overrightarrow{vx}$ has the property vp = vp'. Let γ be a path in the $sector \angle^* xvy$. We use the coordinates $p = (r, \theta)$ where r = pv and $\theta = m \angle^* ovp$ for some base ray \overrightarrow{vo} . The length of γ is $l = \int_{\gamma} \sqrt{\frac{dr^2}{dt} + r^2 \frac{d\theta^2}{dt}} dt$. Let γ' be the image of γ under rotational projection.



FIGURE 3. Rotational Projection

The path length of γ' is given by $l' = \int_{\gamma'} \frac{dr}{dt} dt$. It is easy to see that $l' \leq l$. Thus rotational projection is length reducing for a path network in the sector. Also note that in the case that γ does not have a constant θ value within $sector \angle xvy$ this inequality is strict (l' < l).

For a path γ between two points x and y on a cone, we call $sector \angle xvy$ (or $sector \angle yvx$) a complement sector to γ if it does not contain any points of γ in its interior, or in other words if given any point d such that $d \in sector \angle xvy$, then $\overrightarrow{vd} \cap \{\gamma - \{\overrightarrow{vx}, \overrightarrow{vy}\}\} = \emptyset$.

We let \overline{xy} denote the path that connects x and y with the least length. Note that this path may or may not pass through the vertex. This will be the minimal geodesic from x to y. Similarly a minimal tree with Steiner point at s is a g-minimal tree if it is given by $\overline{as} \cup \overline{bs} \cup \overline{cs}$ (if it is comprised of minimal geodesics). The following proposition describes the relation between minimal geodesics on the cone and complement sectors.

Propositon 3.2. Suppose x and y are points on a given cone distinct from v. Then minimal geodesic \overline{xy} has the following properties:

- (1) There exists a complement sector to \overline{xy} .
- (2) If $m \angle yvx > m \angle xvy$ then $sector \angle yvx$ is the complement sector to \overline{xy} .

Proof. To see (1), suppose that there exists a path γ between the points x and y on some cone, \mathcal{C} , that passes through the interior of both $sector \angle^* xvy$ and $sector \angle^* yvx$. Parameterize the path γ by $\gamma : [0,1] \to \mathcal{C}$ where $\gamma(0) = x$ and $\gamma(1) = y$. Since γ passes through the interiors of both sectors, there must needs be a point $\gamma(t)$ for 0 < t < 1 such that $\gamma(t) \in v\vec{x} \cup v\vec{y}$ and the paths $\alpha = \gamma|_{[0,t]}$ and $\beta = \gamma|_{[t,1]}$ have non-constant θ values. Suppose $\gamma(t) \in v\vec{x}$. Rotationally project α on to $v\vec{x}$ to find α' . Since α has non-constant θ values, this map is strictly length reducing. Thus $\alpha' \cup \beta$ has less length than γ . Rotational projection of β finds a similar result if $\gamma(t) \in v\vec{y}$. Thus γ is not a minimal path, and minimal geodesic \overline{xy} will not pass through the interiors of both $sector \angle^* xvy$ and $sector \angle^* yvx$. The sector whose interior is not traversed will be a complement sector to \overline{xy} .

To see (2), suppose that $m \angle yvx > m \angle xvy$. Let α be an arbitrary path between x and y on the cone such that $\alpha \cap int(sector \angle^* yvx) \neq \emptyset$. We will show that α does not have minimal length. Let x' be the point on the cone opposite x, i.e. x' is a $\psi/2$ rotation of x around the cone. Then $y \in sector \angle xvx'$. Reflect all points of α in sector $\angle^* x' v x$ onto sector $\angle^* x v x'$ such that for point $p \in sector \angle x'vx$ it's reflection $p' \in sector \angle xvx'$ has the property $m \angle pvx = xvx'$ $m \angle xvp'$ and vp = vp'. The image of α under reflection, α' , is entirely contained in sector $\angle xvx'$ and has the same path length as α . Rotationally project all points of α' in sector $\angle^* y v x'$ onto \overrightarrow{vy} . Since this projection is length reducing, the resultant path α'' has length less than α' . Since for every $\alpha \not\subset sector \angle xvy$ there exists $\alpha'' \subset sector \angle xvy$ with shorter length, minimal geodesic \overline{xy} must be contained in sector $\angle xvy$. Thus sector $\angle yvx$ will be the complement sector to \overline{xy} . In the case that $m \angle xy = m \angle yy$ the minimal geodesic need not be unique, and both $sector \angle xvy$ and $sector \angle yvx$ are complement sectors to some minimal geodesic.

Remark 3.3. Note that if $m \angle yvx \ge m \angle xvy$ and $m \angle xvy < 180^\circ$ then \overline{xy} is the map of a straight line segment onto the cone, not containing the vertex. This implies that on a narrow cone no minimal geodesic will contain the vertex since $m \angle xvy \le \psi/2 < 180^\circ$. On a wide cone if $m \angle yvx \ge m \angle xvy \ge 180^\circ$ then $\overline{xy} = \overline{xv} \cup \overline{yy}$.

Similar to the complement sector to a minimal geodesic, we call $sector \angle avb$, $sector \angle bvc$, or $sector \angle cva$ a complement sector to a g-minimal tree on points a, b, and c if that sector contains no point of the tree in its interior. The following proposition extends our result for minimal geodesics to g-minimal trees.

Propositon 3.4. Suppose \mathcal{T} is a g-minimal tree for terminals a, b, and c and s is its Steiner point distinct from v, (\mathcal{T} need not be full). Let \mathcal{K} be the intersection of the complement sectors of the edges of \mathcal{T} . Then

- (1) \mathcal{K} is non-empty,
- (2) \mathcal{K} is one of sector \angle^*avb , sector \angle^*bvc , or sector \angle^*cva , and
- (3) \mathcal{K} is a complement sector to \mathcal{T} .

Proof. Without loss of generality suppose $s \in sector \angle avb$. We will show $sector \angle sva$ is a complement sector to \overline{as} . Suppose not: then $sector \angle avs$ is the complement sector. Thus $m \angle bvs \ge m \angle cvs \ge m \angle avs \ge \psi/2 \ge m \angle sva \ge m \angle svc \ge m \angle svb$. It follows that $sector \angle bvs$ and $sector \angle cvs$ are complement sectors of \overline{bs} and \overline{cs} respectively. Then $\mathcal{K} = sector \angle avs$ and \mathcal{T} is contained in $sector \angle sva$. Rotationally project $\mathcal{T} \cap sector \angle svb$ onto \overline{vb} while fixing $\mathcal{T} \cap sector \angle bva$. The result is a path network connecting a, b, and c with less length than \mathcal{T} . This violates the condition that movement of Steiner points cannot decrease the total length of a minimal tree, and a contradiction is found. Thus the complement sector to \overline{as} is $sector \angle sva$.

bs. Now \overrightarrow{vc} is contained in the complement sector of \overrightarrow{cs} , as well as in $sector \angle sva$ and $sector \angle bvs$. Then \overrightarrow{vc} is a contained in \mathcal{K} , so \mathcal{K} is non-empty, proving (1).

The complement sector to geodesic \overline{cs} is either $sector \angle svc$ or $sector \angle cvs$. Suppose $sector \angle svc$ is its complement sector. Then \mathcal{K} is $sector \angle bvc$. Otherwise the complement sector to \overline{cs} will be $sector \angle cvs$ and \mathcal{K} will be $sector \angle cva$. Then for any g-minimal tree, \mathcal{K} is $sector \angle avb$, $sector \angle bvc$, or $sector \angle cva$. Note that \mathcal{K} could be $sector \angle avb$ in the case where $s \notin sector \angle avb$. This proves (2).

(3) follows from the definition of \mathcal{K} .

Remark 3.5. The proof for Proposition 3.4 supposed that the Steiner point $s \neq v$. The key result however is still true for s = v. Note that for such a tree, similar arguments show that $sector \angle^* avb$, $sector \angle^* bvc$, and $sector \angle^* cva$ are all complement sectors to the tree.

The following proposition for points in the plane is derived from standard theorems of geometry and will be stated without proof. It will be helpful in considering subsequent theorems.

Propositon 3.6. Let A, B, and C be three distinct points in the plane, and suppose $S \in sector \angle ABC$, then $m \angle ABC < m \angle ASC$.

The following discussion extends our results for complement sectors. The key theorem states that g-minimal trees with a given complement sector are unique. This implies that if two g-minimal trees have a common compliment sector, then they must be equivalent. The results of this theorem will be vital to our discussion of Steiner trees on both narrow and wide cones, though in differing application.

Theorem 3.7. All g-minimal trees on a cone with a given complement sector are unique.

To prove this theorem we begin by defining a map of a cone into the plane such that g-minimal trees on the cone map to obstacle avoiding g-minimal trees in the plane. We show that the obstacle avoiding g-minimal trees under consideration are unique, and subsequently we define a one-to-one function between between g-minimal trees on the cone and obstacle avoiding g-minimal trees in the plane. The proof of Theorem 3.7 then follows trivially.

Suppose \mathcal{T} is a g-minimal tree for points a, b, and c on a cone with $sector \angle^* cva$ as a complement sector. We define the *viewpoint map of point* b, Φ_b , to be a map of the cone into the plane defined as follows. Let $\overrightarrow{vb^+}$ be a clockwise rotation of \overrightarrow{vb} by $\overrightarrow{min}\{180^\circ, m\angle^* avb\}$. If $m\angle^* avb > 180^\circ$, rotationally project $sector\angle^* avb^+$ onto $\overrightarrow{vb+}$. Similarly define $\overrightarrow{vb^-}$ a counter-clockwise rotation by $min\{180^\circ, m\angle^* bvc\}$ and rotationally project $sector\angle^* b^- vc$ onto $\overrightarrow{vb^-}$. Seeing that $m\angle^* b^+ vb^- \leq 360^\circ$, we lay out $sector\angle^* b^+ vb^-$ in the plane such that \overrightarrow{vb} coincides with the negative x-axis. We ignore $sector\angle^* b^- vb^+$ since after rotational projection it will contain no point of the tree \mathcal{T} . Let the image of a point p on the cone be labeled P when

mapped into the plane. Label the reflection of the point B in the plane across the y-axis B'.

We denote the image of the cone under Φ_b described above as Λ . Consider the image of \mathcal{T} under the viewpoint map of b, $\Phi_b(\mathcal{T})$. If a point x of \mathcal{T} lies in sector $\angle^* avb$ we consider its image X under the map to be in the upper half plane, even if it lies on $\overrightarrow{VB'}$, and similarly if $y \in sector \angle^* bvc$ then Y is considered in the lower half plane. Under these assumptions $\Phi_b(\mathcal{T})$ will be an obstacle avoiding g-minimal tree for points A, B, and C in the plane where the obstacle is the two sided ray $\overrightarrow{VB'}$. Here a g-minimal tree is obstacle avoiding if no part of the tree crosses through $\overrightarrow{VB'}$, except possibly at the vertex. Note that a g-minimal tree is obstacle avoiding if and only if $sector \angle^* CVA$ is a complement sector to the tree. (See [3] for further information regarding obstacle avoiding trees.)

Lemma 3.8. There is a unique obstacle avoiding g-minimal tree for points A, B, and C in the plane with a two sided ray obstacle as in the setting described above.

Proof. Take any arbitrary obstacle avoiding g-minimal tree \mathcal{Q} on Φ_b with Steiner point S. Construct the planar g-minimal tree on A, B, and C, call it \mathcal{R} , and label its Steiner point S'. This g-minimal tree is unique in the plane. If \mathcal{R} does not pass through $\overline{VB'}$ except possibly at V, then $\mathcal{Q} = \mathcal{R}$ and is uniquely determined. Otherwise, \mathcal{R} crosses $\overrightarrow{VB'}$ on either $\overline{S'A}$ or $\overline{S'C}$. Without loss of generality suppose $\overline{S'C}$ crosses $\overrightarrow{VB'}$. Then $V \in sector \angle BS'C$. Thus $m \angle BVC \geq m \angle BS'C \geq 120^\circ$. It follows that $S \notin sector \angle BVC$. Since no point of \mathcal{Q} is contained in the interior of $sector \angle CVA$, then $S \in sector \angle AVB$. Also since $m \angle AVB \leq 180^\circ$, neither \overline{SA} nor \overline{SB} pass through the vertex. Now since \mathcal{R} passes through $\overline{VB'}$, there does not exist a obstacle avoiding q-minimal tree in Λ that does not contain V. Thus $V \in \overline{SC}$. This implies that \mathcal{Q} includes \overline{VC} and the remainder of \mathcal{Q} is a planar q-minimal tree on points A, B, and V. The uniqueness of this tree in the plane implies that \mathcal{Q} is uniquely determined. Since \mathcal{Q} is uniquely determined in both cases, this implies that bounded q-minimal trees are unique on Λ .

We define the function $\varphi_b(\mathcal{T})$ that maps trees on points a, b, and c that are g-minimal in $sector \angle *avc$, with complement sector $sector \angle *cva$, to obstacle avoiding g-minimal trees on points A, B, and C in Λ . Let $\varphi_b(\mathcal{T})$ be the image of g-minimal tree \mathcal{T} under the map Φ_b . This function is well defined since the image of a g-minimal tree under Φ_b is a obstacle avoiding g-minimal tree.

Lemma 3.9. $\varphi_b(\mathcal{T})$ is a one-to-one function.

Proof. Since $\varphi_b(\mathcal{T})$ is well defined, we need only show that any obstacle avoiding g-minimal tree in Λ has a unique inverse image. Let \mathcal{Q} be an arbitrary obstacle avoiding g-minimal tree in Φ_b . Note that the restriction of Φ_b to interior of

sector $\angle^* b^+ vb^-$ on the cone is an isometry. If $m \angle^* avb < 180^\circ$ and $m \angle^* bvc < 180^\circ$ this implies that \mathcal{Q} has a unique inverse image on the cone. Suppose instead that $m \angle^* avb \ge 180^\circ$. Then a is mapped by Φ_b onto $\overrightarrow{VB'}$. Then by the construction of obstacle avoiding g-minimal trees found in Lemma 3.8, $\mathcal{Q} \cap \overrightarrow{VB'}$ in the upper half plane must be \overrightarrow{VA} . If we assert that the inverse image of \mathcal{Q} must be a g-minimal tree on the cone, \overrightarrow{VA} has the unique inverse image \overrightarrow{va} on the cone. Similarly for the lower half plane if $m \angle^* bvc \ge 180^\circ$. Since both portions, $int(sector \angle^* B'VB')$ and $\overrightarrow{VB'}$, have a unique inverse image under $\varphi_b(\mathcal{T}), \varphi_b(\mathcal{T})$ is a one-to-one function. \Box

We now have all the tools necessary to complete the proof of Theorem 3.7. Proof of Theorem 3.7. To show the uniqueness of g-minimal trees with a given complement sector, suppose there are two such trees \mathcal{T}_1 and \mathcal{T}_2 on a given cone with the same complement sector. Without loss of generality suppose that this sector is sector $\angle^* cva$. Since obstacle avoiding g-minimal trees are unique in Λ , $\varphi_b(\mathcal{T}_1) = \varphi_b(\mathcal{T}_2)$. But since $\varphi_b(\mathcal{T})$ is one-to-one, this implies that $\mathcal{T}_1 = \mathcal{T}_2$. Thus g-minimal trees with a given complement sector are unique.

Lemma 3.10. Let a g-minimal tree on an oriented triple of points a, b, and c with Steiner point at s be given with sector $\angle^* cva$ a complement sector. If the tree does not pass through the vertex then,

- (1) $m \angle avc < 240^{\circ}$ and
- (2) $m \angle avb < 180^{\circ}$ and $m \angle bvc < 180^{\circ}$.

Proof. Since $sector \angle^* cva$ a complement sector to the tree then $s \in sector \angle^* avc$. Now since \overline{as} and \overline{cs} do not pass through the vertex, we note that $m \angle^* avc < m \angle^* asc = 240^\circ$ by Proposition 3.6, proving (1).

If the g-minimal tree is degenerate and is equal to $\overline{ab} \cup \overline{bc}$, then both of these minimal segments do not contain the vertex, and $m\angle^*avb$ and $m\angle^*bvc$ are less than 180°. Otherwise a degenerate g-minimal tree would contain the minimal geodesic \overline{ac} which does not contain the vertex, so $m\angle^*avb \leq m\angle^*avc < 180^\circ$. Similarly $m\angle^*bvc < 180^\circ$. If the g-minimal tree is non-degenerate suppose $m\angle^*bvc \geq 180^\circ$. Now $s \notin sector\angle^*avb$, since otherwise $m\angle^*svc > 180^\circ$ and \overline{sc} would contain the vertex. Thus $s \in sector\angle^*bvc$. But that would imply by Proposition 3.6 that $180^\circ < m\angle^*bvc < m\angle^*bsc = 120^\circ$, a contradiction. Therefore $m\angle^*bvc$, and similarly $m\angle^*avb$, must be less than 180° , showing (2).

Remark 3.11. Lemma 3.10 states necessary conditions for a g-minimal tree that does not pass through the vertex. These are not however conditions sufficient to conclude that a g-minimal tree does not pass through the vertex. Sufficient conditions are complex and somewhat ambiguous, though for a non-degenerate tree it is sufficient to show that when the e-point is constructed opposite \overline{ab} from c the Simpson line does not contain the vertex, and for a degenerate tree to show

that the shorter of \overline{ac} and \overline{bc} does not contain the vertex. In this case the *g*-minimal tree will not pass through the vertex. The exact conditions follow with proof omitted:

- (1) Suppose $m \angle bac \ge 120^{\circ}$ or $m \angle acb \ge 120^{\circ}$. The tree will not contain the vertex if and only if $m \angle avc < 180^{\circ}$.
- (2) Suppose $m \angle cba \ge 120^\circ$. Conditions (1) and (2) from Lemma 3.10 are sufficient to show that the tree does not contain the vertex.
- (3) Suppose the g-minimal tree is non-degenerate. The tree will not contain the vertex if and only if

$$m\angle^{*}vcb + m\angle^{*}cba + \arcsin\left[\frac{\sin(m\angle^{*}vba + 60^{\circ})(vb)}{\sqrt{(vb)^{2} + (ab)^{2} - 2(ab)(vb)\cos(m\angle^{*}vba + 60^{\circ})}}\right] > 120^{\circ}.$$

4. The Three Point Steiner Problem on Narrow Cones

In this section we will develop an algorithm that will solve the three point Steiner problem on a narrow cone. Our algorithm transforms the three point problem on the cone to the three point problem on the plane. This is done by cutting along \vec{vx} on the cone, where v is the vertex of the cone and x is a point on the cone distinct from v, and laying the cone flat in the plane. When laid out in the plane, the surface of the cone will cover all but a sector of the plane. The portion of the plane not covered by the laid out cone we will call the *cut wedge*.

We first show that the Steiner minimal tree connecting three given points will be contained within the region of the laid out cone for a cut along an appropriate ray. We will then present an algorithm for cutting a cone to find and compare all possible g-minimal trees, thus isolating the Steiner minimal tree.

4.1. Three Candidate *g*-Minimal Trees. In subsequent proofs we require a method of cutting the cone to use the geometric properties of the plane, as mentioned above. This will be done more specifically in the following manner:

A ray \overrightarrow{vx} will be given. Cut the cone along \overrightarrow{vx} . Lay the cone flat in the plane with vertex v at the origin and the cut wedge centered on the positive x-axis. Let \mathcal{R} be the region in the plane that is covered by the laid out cone after cutting. We will label the points in \mathcal{R} as follows: if p is a point on the cone that does not lie on \overrightarrow{vx} then its corresponding point in \mathcal{R} is unique and we will label that point in \mathcal{R} as P. If p lies on \overrightarrow{vx} then there are two points in \mathcal{R} on the border of the cut wedge that correspond to p, one in the upper half plane and one in the lower half plane. Label the point in the upper half plane P and the point in the lower half plane P'. (See Figure 4)

Lemma 4.1. Suppose \mathcal{T} is a g-minimal tree for given points a, b, and c on a narrow cone with a given complement sector. If the cutting method is applied to the cone for a cut along any ray \overrightarrow{vd} where d is in the complement sector, then the image of \mathcal{T} in the laid out cone will be a Steiner minimal tree for the points A, B, and C in the plane.



FIGURE 4. The Cone in the Plane

Proof. This follows from Lemma 3.4. Without loss of generality let $sector \angle construction construction is defined and the generality of t$

By Theorem 3.7 we note that there are at most three g-minimal trees for three points on a narrow cone, one corresponding to each of the three possible complement sectors. Any g-minimal tree must have a complement sector and thus be equivalent to one of these three. Proposition 3.1 shows us that there must be at least one g-minimal tree, the Steiner minimal tree. Unfortunately we cannot be certain that given one of $sector \angle^* avb$, $sector \angle^* bvc$, or $sector \angle^* cva$ that there exists a g-minimal tree with that sector as its complement. For example, suppose that when the cone is laid out in the plane by a cut in $sector \angle^* avb$, the g-minimal tree connecting points A, B, and C in the plane passes through the cut wedge. Then a g-minimal tree with $sector \angle^* avb$ as its complement sector is not constructible on the cone. **Remark 4.2.** It is useful to note that if a g-minimal tree exists with a given complement sector, it may be laid out in the plane by a cut along any ray in that sector. This includes the rays on its boundary. Note then that $\overrightarrow{va} \subseteq sector \angle^* avb$ and $\overrightarrow{va} \subseteq sector \angle^* cva$. Thus since \overrightarrow{va} is in two different complement sectors, a cut along \overrightarrow{va} will yield two g-minimal trees contained in \mathcal{R} (assuming both trees are constructible). In this case the tree contained in $sector \angle^* avc$ will be mapped to a g-minimal tree in \mathcal{R} on the points A, B, and C. The tree contained in $sector \angle^* bva$ will be mapped in \mathcal{R} to a g-minimal tree on points A', B, and C. Thus cutting along the rays $\overrightarrow{va}, \overrightarrow{vb}$, and \overrightarrow{vc} will allow us to compare two g-minimal trees in the same plane. We will use this tool in the next section as we describe the cutting algorithm that allows us to compare g-minimal trees and isolate the Steiner minimal tree.

4.2. Cutting Algorithm. Given a narrow cone with vertex v and distinct points a, b, and c on the cone also distinct from v, our cutting algorithm will proceed as follows:

Since there are at most three possible g-minimal trees on the narrow cone, let the g-minimal tree formed in the plane by a cut in $sector \angle^* avb$ be denoted \mathcal{T}_1 . Note that all cuts in this sector yield congruent g-minimal trees. Similarly denote the tree formed by a cut in $sector \angle^* bvc$ as \mathcal{T}_2 and the tree formed by a cut in $sector \angle^* cva$ as \mathcal{T}_3 . Cut along \vec{va} . Because $\vec{va} \subseteq sector \angle^* avb$ and $sector \angle^* cva$, there are two possible g-minimal trees constructible in the laid out cone; \mathcal{T}_1 and \mathcal{T}_3 . Construct these two g-minimal trees in the plane (if possible) and compare their total length. If at any time one of the trees for comparison is not constructible, it may be ignored. If \mathcal{T}_1 has length less than or equal to \mathcal{T}_3 make a new cut along \vec{vb} , otherwise make the new cut along \vec{vc} . Note that the shorter of the \mathcal{T}_1 and \mathcal{T}_3 will again be represented in the laid out cone with respect to the new cut. Thus we will similarly be able to compare the shorter tree of the first two with \mathcal{T}_2 , the last of the three candidates for the Steiner minimal tree. The shorter of these two remaining trees will be the Steiner minimal tree.

In the case that one of the points a, b, or c coincides with the vertex, the solution greatly simplifies. Without loss of generality suppose that c = v and that $m \angle avb \leq m \angle bva$. The Steiner minimal tree must then be contained in $sector \angle avb$, since reflection followed by rotational projection is length reducing for any tree that exits $sector \angle avb$, as in Lemma 3.4. Thus there is only one possible complement sector, $sector \angle bva$. Cutting along va and laying out the cone in the plane will allow you to identify the only g-minimal tree possible, which will be the Steiner minimal tree.

Example 4.3. Suppose points a, b, and c on a narrow cone of $\psi = 300^{\circ}$ are given such that cutting along \overrightarrow{va} yields the laid out cone on the left in Figure 5. Constructing the two g-minimal trees (one using point A, the other using A') shows that the g-minimal tree contained in $sector \angle AVC$, on points A, B, and C, is shorter than the tree in $sector \angle BVA'$, on points A', B, C. If both



FIGURE 5. The cutting algorithm

trees were non-degenerate, we could compare lengths of the two Simpson lines, but since one is degenerate, we compare its total length to the other tree. Since the tree contained in $sector \angle AVC$ has total length less than the tree contained in $sector \angle BVA'$, our next cut will be along \overrightarrow{vc} , as on the right in Figure 5. The tree contained in $sector \angle AVC$ in the previous laid out cone is congruent to the tree contained in $sector \angle AVC$ in the new laid out cone. When the two g-minimal trees are constructed in the new laid out cone however, we notice that the g-minimal tree for points A, B, and C' crosses the cut wedge, and is not contained in $sector \angle C'VB$. This g-minimal tree is thus not constructible on the cone, and may be ignored. Then the g-minimal tree contained in $sector \angle AVC$, on points A, B, and C, has the shortest total path length of all g-minimal trees on any laid out cone. When this tree is mapped back onto the cone, it will be the Steiner minimal tree for points a, b, and c on the cone.

We have here shown that the three point Steiner problem on narrow cones may be solved by laying out the cone in two cuts and comparing three candidate positions for the Steiner minimal tree. However, this method is not possible on the wide cone since the wide cone cannot be laid out in the plane. The following section will deal with this case and present algorithms that solve the three point Steiner problem on wide cones.



FIGURE 6. Gq-minimal trees of Types III, II, and I

5. The Three Point Steiner Problem on Wide Cones

In this section we consider the Steiner problem for three points a, b, and c on a wide cone. The following proves that there is only one g-minimal tree for any three points on a given wide cone. Since there is only one g-minimal tree on a wide cone, this must also be the Steiner minimal tree. Algorithms that identify this g-minimal tree are presented subsequently.

5.1. Uniqueness of g-Minimal Trees on Wide Cones. At this point it is useful to define a classification of g-minimal trees on wide cones on three points a, b, and c distinct from the vertex v. A g-minimal tree on a wide cone for which the Steiner point lies on the vertex will be called *type III*. A g-minimal tree in which the interior of one edge of the tree meets the vertex will be called *type II*. Lastly, a g-minimal tree that does meet the vertex will be called *type I*. Note that types II and I also include degenerate g-minimal trees for which the Steiner point is considered to coincide with one of the three terminals. See Figure 6.

Before we prove uniqueness it will be advantageous to describe some properties of g-minimal trees of type III, II, and I. Note that Lemma 3.10 applies specifically to g-minimal trees of type I. (All g-minimal trees for three points distinct from the vertex on narrow cones are of type I.) In particular let \mathcal{T} be a g-minimal tree of type I for points a, b, and c on the cone such that complement sector of \mathcal{T} is $sector \angle^* cva$. Then $m \angle^* avc < 240^\circ$, $m \angle^* avb < 180^\circ$, and $m \angle^* bvc < 180^\circ$. The following two lemmas deal with g-minimal trees of types II and III respectively, proving comparable results for each.

Lemma 5.1. Suppose \mathcal{T} is a g-minimal tree of type II with Steiner point s such that $s \in \operatorname{sector} \angle^* \operatorname{avb}$. Then

- (1) $m \angle avb < 120^\circ$,
- (2) $m \angle bvc > 120^{\circ}$, and $m \angle cva > 120^{\circ}$.

Proof. (1) follows directly from Lemma 3.6: $m \angle avb < m \angle asb = 120^{\circ}$.

Note that $v \in \overline{sc}$. Since $m \angle bsv \ge 120^\circ$, it follows that $m \angle svb < 60^\circ$. Since minimal geodesic \overline{sc} passes through the vertex, $m \angle svc \ge 180^\circ$. Thus $m \angle bvc = m \angle svc - m \angle svb > 180^\circ - 60^\circ = 120^\circ$. Similarly $m \angle cva > 120^\circ$, proving (2).

Lemma 5.2. Suppose \mathcal{T} a g-minimal tree of type III. Then $m \angle avb \geq 120^{\circ}$, $m \angle bvc \geq 120^{\circ}$, and $m \angle cva \geq 120^{\circ}$.

Proof. This follows directly from the properties of minimal trees. Otherwise, a small movement of the Steiner point in the direction of the narrower sector would shorten the tree. \Box

One last note of interest is that g-minimal trees on a wide cone may have more than one complement sector. While Proposition 3.4 states that every g-minimal tree connecting three points on a cone has at least one complement sector, trees of types II and III have two and three complement sectors respectively. We will utilize this fact in our proof of uniqueness using Theorem 3.7.

We now prove the uniqueness of g-minimal trees on a wide cone.

Theorem 5.3. There exists one and only one g-minimal tree for three points on a given wide cone.

Proof. For points a, b, and c on a given wide cone, by Lemma 3.1 the Steiner minimal tree exists with Steiner point at s, possibly coinciding with a, b, c, or v. It is helpful to note that as a consequence of Lemma 3.4 and the above classifications, there are exactly seven possible configurations for a g-minimal tree on a wide cone: one of type III (where s = v), three of type II (where $s \in sector \angle^* avb, sector \angle^* bvc,$ or $sector \angle^* avc, sector \angle^* bva,$ or $sector \angle^* cvb$). We will prove the uniqueness of g-minimal trees on wide cones by considering the type of the Steiner minimal tree and eliminating all other possible configurations.

Case I: Suppose the Steiner minimal tree is of type I. Without loss of generality suppose that its complement sector is $sector \angle^* cva$. By Theorem 3.7 any g-minimal tree with this sector as its complement is identical to the Steiner minimal tree. The three configurations that fail to meet this condition are a tree of type II with Steiner point contained in $sector \angle^* cva$ and trees of type I contained in $sector \angle^* bva$ or $sector \angle^* cvb$. Since by Lemma 3.10 $m \angle^* avc < 240^\circ$, on the wide cone $m \angle^* cva = \psi - m \angle^* avc > 360^\circ - 240^\circ = 120^\circ$. Thus by Lemma 5.1 there can be no g-minimal tree of type II with its Steiner point contained in $sector \angle^* cva$. Now suppose there is a g-minimal tree, \mathcal{T} , of type I contained in $sector \angle^* bva$. This implies that $m \angle^* avb > 120^\circ$ and $m \angle^* bvc < 120^\circ$ by applying Lemma 3.10 to both the Steiner minimal tree and \mathcal{T} . Since both $m \angle^* avb > 120^\circ$ and $m \angle^* cva > 120^\circ$, the Steiner point of both trees must be contained in $sector \angle^* bvc$.

Suppose one of these trees is full, and without loss of generality, let this be the Steiner minimal tree. The uniqueness of minimal geodesics on the wide cone implies that the Simpson line \overline{ea} is unique and cannot be contained in both $sector \angle^* avc$ and $sector \angle^* bva$ without containing the vertex. Thus \mathcal{T} cannot be full. In addition, \overline{ea} passing through bc implies that $m \angle ace < 180^\circ$. But $m \angle acb = m \angle ace - 60^\circ < 120^\circ$, and similarly $m \angle cba < 120^\circ$. If \mathcal{T} is degenerate it must have its Steiner point in sector $\angle bvc$ coincide with either b or c, but since the angles formed at those terminals are less than 120° , neither will be a g-minimal tree. Because of this contradiction neither tree can be full. Suppose instead that both trees are degenerate. In this case each tree is equal to the union of the shortest two of the segments ab, bc, and \overline{ac} . This also leads to a contradiction since these are identical and cannot be contained in two distinct sectors. Thus there cannot be another Steiner tree of type I contained in sector $\angle *bva$ nor similarly in sector $\angle *cvb$. The Steiner minimal tree proves to be the unique q-minimal tree on the wide cone.

- **Case II:** Suppose the Steiner minimal tree is of type II. Without loss of generality, suppose $s \in sector \angle avb$. This implies that $v \in \overline{sc}$, and $sector \angle bvc$ and $sector \angle cva$ are both complement sectors of the Steiner minimal tree. All *g*-minimal trees that share either of these complement sectors are identical to the Steiner minimal tree by Theorem 3.7. The only one of the seven possible configurations of a *g*-minimal tree on the cone that does not meet this condition is a tree of type I with complement sector $sector \angle avb$. By Lemma 5.1, $m\angle avb < 120^{\circ}$ and $m\angle bva = \psi m\angle avb > 360^{\circ} 120^{\circ} = 240^{\circ}$. Thus by Lemma 3.10 there can exist no *g*-minimal tree of type I contained in $sector \angle bva$, and the Steiner minimal tree is the unique *g*-minimal tree on the cone.
- **Case III:** Suppose the Steiner minimal tree is of type III (s = v). Since any other *g*-minimal tree must have a complement sector in common with the Steiner minimal tree, Theorem 3.7 implies that the Steiner minimal tree is the unique *g*-minimal tree on the cone, as all other *g*-minimal trees must be identical to it.

5.2. Isolating g-minimal Trees on Wide Cones. In this section we will describe constructions for g-minimal trees on the wide cone. The constructed tree in each case will be the Steiner minimal tree because of the uniqueness of g-minimal trees proven in the previous section.

Let a, b, and c be an oriented triple of points distinct from the vertex. We consider the angles $\angle avb$, $\angle bvc$, and $\angle cva$. There are three cases, namely: (1) all angles have measure $\ge 120^{\circ}$, (2) exactly two angles have measure $\ge 120^{\circ}$, and (3) exactly one angle has measure $\ge 120^{\circ}$. Note that on the wide cone there must be at least one angle that has measure $\ge 120^{\circ}$. Theorems 5.4, 5.5, and 5.7 will

address cases (1), (2), and (3) respectively. Theorem 5.8 will then address the case where one of the terminals a, b, or c lies on the vertex.

Theorem 5.4. Suppose a, b, and c are an oriented triple of points on a wide cone distinct from the vertex such that $m \angle avb \ge 120^\circ$, $m \angle bvc \ge 120^\circ$, and $m \angle cva \ge 120^\circ$. Then the Steiner minimal tree for points a, b, and c is of type III and is equivalent to $\overline{av} \cup \overline{bv} \cup \overline{cv}$.

Proof. We note by the angle conditions found in Lemma 3.10 and Lemma 5.1 that no *g*-minimal tree of types II or I may exist on this wide cone. Since the Steiner minimal tree must exist by Proposition 3.1, the Steiner minimal tree must be of type III and equivalent to $\overline{av} \cup \overline{bv} \cup \overline{cv}$.

Theorem 5.5. Suppose a, b, and c are an oriented triple of points on a wide cone distinct from the vertex such that $m \angle *avb < 120^\circ$ and both $\angle *bvc$ and $\angle *cva$ have measure greater than or equal to 120° . Then the Steiner minimal tree will be of type II or I and its Steiner point will be contained in sector $\angle *avb$. Moreover if the Steiner tree is degenerate, it will include \overline{ab} .

Proof. Since $m\angle^*avb < 120^\circ$, Lemma 5.2 implies that there are no g-minimal trees of type III on the cone. Also $m\angle^*bvc > 120^\circ$ and $m\angle^*cva > 120^\circ$ imply by Lemma 5.1 that no g-minimal tree of type II exists with Steiner point contained in $sector\angle^*bvc$ or $sector\angle^*cva$. By Lemma 3.10, there is no g-minimal tree of type I contained in $sector\angle^*bva$. Thus the Steiner minimal tree must be either type II with Steiner point contained in $sector\angle^*avb$ or of type I contained in $sector\angle^*avb$. Note that in any case the Steiner point must be contained in $sector\angle^*avb$. Thus in the degenerate case s = a or s = b. This implies that if the Steiner minimal tree is degenerate, it must include the edge \overline{ab} .

The following cases give conditions for the delineation of types II and I. The g-minimal tree found will be the Steiner minimal tree due to uniqueness Theorem 5.3. To get started construct the e-point of \overline{ab} opposite c. If this e-point does not exist, the tree is of type I contained in the triangle $\overline{ab} \cup \overline{bc} \cup \overline{ac}$. Construct the geodesic \overline{ec} . (This will be $\overline{ev} \cup \overline{vc}$ if $m \angle evc$ and $m \angle evec$ are $\geq 180^{\circ}$)

- **Case I:** \overline{ec} does not intersect \overline{ab} . Construct \overline{ac} and \overline{bc} . Without loss of generality, suppose that \overline{ac} has a shorter total length.
 - **Case A:** \overline{ac} does not pass through v. The Steiner minimal tree is degenerate of type I and is $\overline{ab} \cup \overline{ac}$.
 - **Case B:** \overline{ac} passes through $v \ (m \angle avc \text{ and } m \angle cva \ge 180^\circ)$. The Steiner minimal tree is degenerate of type II and is $\overline{ab} \cup \overline{av} \cup \overline{vc}$.

Case II: \overline{ec} intersects ab.

Case A: \overline{ec} does not pass through v. The Steiner minimal tree is a full tree of type I with line segments $\overline{as} \cup \overline{bs} \cup \overline{cs}$, where $s \in \overline{ec}$ is the Steiner point found using the *e*-point construction on points a, b, and c.



С

FIGURE 7. Cases for Theorem 5.5

Case B: \overline{cc} passes through $v \ (m \angle evc \text{ and } m \angle evc \ge 180^\circ)$. The Steiner minimal tree is a full Steiner tree of type II with line segments $\overline{as} \cup \overline{bs} \cup \overline{sv} \cup \overline{vc}$, where $s \in \overline{ev}$ is the Steiner point found using the *e*-point construction on points a, b, and v. Note that $s \notin \overline{vc}$ since $s \in sector \angle avb$.

Remark 5.6. Note here that the above argument utilizes the same conditions mentioned in Remark 3.11. The more exact conditions mentioned therein will delineate between trees of type II and I without the need of an *e*-point construction.

Theorem 5.7. Suppose a, b, and c are an oriented triple of points on a wide cone distinct from the vertex such that $m \angle cva \ge 120^\circ$ and both $\angle avb$ and $\angle bvc$ have measure less than 120°. Then the Steiner minimal tree is of type I contained in sector $\angle avc$.

Proof. Since $m \angle avc < 240^\circ$, we lay $sector \angle avc$ out in the plane. Construct the *g*-minimal tree on points A, B, and C in the plane, call it \mathcal{T} , with Steiner point at S. It is sufficient to show that \mathcal{T} is contained in $sector \angle AVC$.

First suppose \mathcal{T} is full. We now show that $\overline{sa} \subset \operatorname{sector} \angle AVE$ and does not contain the vertex. See Figure 8. Since \mathcal{T} is full, the Simpson line \overline{EA} crosses \overline{BC} . Note V cannot be inside the circle circumscribing $\triangle BCE$, since $m\angle BVC < 120^\circ$. (The angle between B and C from any point on the boundary of this circle has measure 120° while any point on the interior must have measure greater than 120° by Lemma 3.6.) Let M be the intersection of \overline{VB} with the circle and N be the intersection of \overline{VE} with the circle. Since $\angle BVE$ cuts the



FIGURE 8. Construction for Theorem 5.7

circle with two secant lines,

$$m \angle BVE = \frac{1}{2} (m(\stackrel{\frown}{BE}) - m(\stackrel{\frown}{MN})) < \frac{1}{2} (120^{\circ}) = 60^{\circ}.$$

Therefore, $m\angle AVE = m\angle AVB + m\angle BVE < 120^\circ + 60^\circ = 180^\circ$. Since $m\angle AVE < 180^\circ$, $\overline{EA} \subset sector\angle AVC$ and does not pass through the vertex. Thus $\overline{SA} \subset \overline{EA}$ does not pass through the vertex and is contained in $sector\angle AVC$. Similarly $\overline{SC} \subset sector\angle AVC$ and does not pass through the vertex. Now both $sector\angle AVB$ and $sector\angle AVC$ are convex and \overline{SB} must be contained in one of these sectors. This implies that $\overline{SB} \subset sector\angle AVC$. Since S is not at the vertex, \overline{SB} does not pass through the vertex. These results imply that \mathcal{T} is contained in $sector\angle AVC$ and does not contain the vertex.

Suppose instead that \mathcal{T} is degenerate. Either $\mathcal{T} = \overline{AB} \cup \overline{BC}$, $\mathcal{T} = \overline{AB} \cup \overline{AC}$, or $\mathcal{T} = \overline{AC} \cup \overline{BC}$. Note that \overline{AB} and \overline{BC} are contained in $sector \angle^* AVC$ and do not contain the vertex since $m \angle^* AVB < 120^\circ$ and $m \angle^* BVC < 120^\circ$. Thus if $\mathcal{T} = \overline{AB} \cup \overline{BC}$, then \mathcal{T} is contained in $sector \angle^* AVC$ and does not contain the vertex. Suppose instead that $\mathcal{T} = \overline{AB} \cup \overline{AC}$. Then $m \angle^* BAC \ge$ 120° . Thus $m \angle^* AVB < 180^\circ - m \angle^* BAV < 180^\circ - m \angle^* BAC < 60^\circ$. Now $m \angle^* AVC = m \angle^* AVB + m \angle^* BVC < 60^\circ + 120^\circ = 180^\circ$ implies that \overline{AC} is contained in $sector \angle^* AVC$ and does not contain the vertex. Thus \mathcal{T} is contained in $sector \angle^* AVC$ and does not contain the vertex. The same result follows from a similar argument if we suppose instead that $\mathcal{T} = \overline{AC} \cup \overline{BC}$.

In in any of the above cases, \mathcal{T} is contained in $sector \angle AVC$ and does not contain the vertex. Thus when \mathcal{T} is mapped back onto the cone, the result is a *g*-minimal tree of type I contained in $sector \angle avc$. Uniqueness implied by Theorem 5.3 shows that this is the Steiner minimal tree. \Box

The next theorem deals with the case where one of a, b, or c coincide with the vertex. Since the vertex must always be contained in any g-minimal tree on these points, classification into types III, II, or I no longer apply.

Theorem 5.8. Suppose a and b are an oriented pair of points on a wide cone having vertex v and $m \angle avb \leq m \angle bva$. The Steiner minimal tree for a, b, and v is contained in sector $\angle avb$, and if $m \angle avb \geq 120^{\circ}$, then the Steiner minimal tree for a, b, and v is degenerate and equal to $\overline{av} \cup \overline{bv}$.

Proof. Note that $sector \angle bva$ is a complement sector to the Steiner minimal tree. Otherwise the tree could be shortened by arguments similar to Propositions 3.2 and 3.4. Thus the Steiner minimal tree is contained in $sector \angle avb$. Suppose $m\angle avb \ge 120^\circ$. This angle condition implies that the Steiner minimal tree must be degenerate with Steiner point at v. (See [2, 6]) Thus the Steiner minimal tree is equal to $\overline{av} \cup \overline{bv}$ in this case.

6. The n-point problem

For n > 3, the *n*-point Steiner problem on the cone has additional complexity, the same as the *n*-point problem in the plane. In particular, the number of possible combinatorial structures that may realize a *g*-minimal tree increases as *n* increases. Recall that there are at most n - 2 Steiner points in a *g*-minimal tree. Thus the only possible combinatorial structures for a *g*-minimal tree on three points is three segments meeting at a distinct Steiner point in the full case or two edges of the triangle formed by the three points in the degenerate case. However, as illustrated in Fig. 9, for n > 3 there are many more possible combinatorial configurations.

Definition 6.1. Suppose that \mathcal{T}_1 and \mathcal{T}_2 are trees connecting a given set of n terminals in a surface S and $X \subset S$. We say that \mathcal{T}_1 and \mathcal{T}_2 have the same Steiner topology relative to X, denoted $\mathcal{T}_1 \simeq \mathcal{T}_2$ rel X, if:

- (1) For i = 1, 2, there are embeddings $\xi_i : \mathcal{T} \to \mathcal{T}_i$ for some abstract tree \mathcal{T} .
- (2) There is an isotopy $H : \mathcal{T} \times I \to \mathcal{S}$ between ξ_1 and ξ_2 that fixes the terminals of \mathcal{T} and satisfies $H(\mathcal{T} \times (0, 1)) \subset \mathcal{S} X$.

The collection of all trees in the surface having the same Steiner topology as \mathcal{T}_1 is denoted $[\mathcal{T}_1]$ and is called the *Steiner topology class of* \mathcal{T}_1 .

Recall that a g-minimal tree is a local minimizer, i.e., the tree is comprised of minimal length segments and the length of the tree cannot be shortened by small movements of the Steiner points. To identify the Steiner minimal tree, one should consider all possible Steiner topologies for trees connecting the given nterminals that can be realized as a g-minimal tree. The shortest amongst these g-minimal trees is the Steiner minimal tree.

The minimal path network connecting n points, without allowing the addition of vertices, is called a *minimal spanning tree*. Note that any degenerate g-minimal



FIGURE 9. Full Steiner topologies for six points.

tree is the union of minimal spanning trees and full g-minimal trees on subsets of the given terminals. Thus the constructibility a configuration can be broken down to the constructibility of a collection of minimal spanning trees and full g-minimal trees.

The minimal spanning tree connecting a given set of terminals $\{v_{i1}, v_{i2}, \ldots, v_{ik}\}$ is always realized on surfaces for which minimal segments between any two points exists. Constructing the minimal spanning tree is a fairly simple matter using Prim's Algorithm. Choose one of the the points. Find the point closest to it and construct a minimal segment between the two points. Now two points have been selected. Given j selected points, find a point, not yet selected, that is closest to the selected set. Construct a minimal segment connecting this point to the point in the selected set closest to it. The minimal spanning tree is the union of segments constructed after all points have been selected.

A full g-minimal tree connecting a given set a terminals having a given Steiner topology is not always realizable. However, if it can be realized the construction can be reduced to a series of three point problems. Here we will provide some details as to how Melzak's algorithm in the plane can be applied to construct a full g-minimal tree having a prescribed Steiner topology for an n-point Steiner problem on a cone.

6.1. Melzak's Algorithm. Let $\mathcal{P}_0 = \{A_1, A_2, \ldots, A_n\}$ be the given set of terminals and \mathcal{T}_0 a given combinatorial configuration on \mathcal{P}_0 . It is desired to construct a full *G*-minimal tree having the combinatorial configuration of \mathcal{T}_0 , if possible. There are two phases to the construction: the de-construction phase and the reconstruction phase.

The De-Construction Phase

It is a basic fact that every tree has at least two pairs of siblings. Without loss of generality suppose that A_1 and A_2 are siblings in \mathcal{T}_0 . Let E_1 be an *e*-point for A_1 and A_2 . Let \mathcal{T}_1 be the configuration obtained by replacing the combinatorial carrot (\wedge) in \mathcal{T}_0 , designating A_1 and A_2 as siblings, with a vertex labelled E_1 . Let $\mathcal{P}_1 = \{B_1, B_2, \ldots, B_{n-1}\}$ be the set of points $\{E_1, A_3, \ldots, A_n\}$ ordered so that B_1 and B_2 are siblings in \mathcal{T}_1 . We will define \mathcal{P}_{k+1} and \mathcal{T}_{k+1} for $k = 1, \ldots, n-2$ by induction. At the *k*th stage suppose that $\mathcal{P}_k = \{X_1, X_2, \ldots, X_{n-k}\}$ and \mathcal{T}_k are given so that X_1 and X_2 are siblings in \mathcal{T}_k . Let \mathcal{T}_{k+1} be the configuration obtained by replacing the combinatorial carrot in \mathcal{T}_k , designating X_1 and X_2 as siblings, with a vertex labelled E_{k+1} . Let $\mathcal{P}_{k+1} = \{Y_1, Y_2, \ldots, Y_{n-k-1}\}$ be the set of points $\{E_{k+1}, X_3, \ldots, X_{n-k}\}$ ordered so that Y_1 and Y_2 are siblings in \mathcal{T}_{k+1} , when k < n-2. (Note that if k = n-2, then \mathcal{T}_{k+1} consists of two points.)

The Re-Construction Phase

The set \mathcal{T}_{n-1} contains precisely two points. Connect these two points by a line segment and label the segment Γ_{n-1} . (This initial line segment is called a *Simpson line*. The length of the Simpson line is the same as the length of

the g-minimal tree that will result at the end of the reconstruction phase, if constructible.) For j = n - 2, ..., 1, let F_{j+1} be a vertex of Γ_{j+1} that is connected to E_{j+1} by a single segment σ_{j+1} . (Note that $\sigma_{n-1} = \Gamma_{n-1}$.) Let X_1 and X_2 be the points in the de-construction phase that where replaced by E_{j+1} at the *j*th stage. Let τ_j be the solution to the three point Steiner problem on $\{X_1, X_2, F_{j+1}\}$. If τ_j is not full, then the desired construction is not possible. If τ_j is full, define $\Gamma_j = (\Gamma_{j+1} - \overline{E_{j+1}F_{j+1}}) \cup \tau_j$. If the construction is successful at every stage, then the tree Γ_1 is a g-minimal tree with the desired combinatorial structure.

6.2. The *n*-point problem on narrow and wide cones. In order to generalize the work done in the previous sections, we consider an alternative, but equivalent, point of view.

To set the scenery, let \mathcal{C} be a cone, either narrow or wide, with cone angle ϕ , vertex v, and designated ray \overrightarrow{vx} . We assign each point $p \in \mathcal{C} - \{v\}$ coordinates $[\rho, \theta]$ where ρ is the distance from p to v and $\theta = m \angle^* x v p$. In the case that p = v, then we assign $\rho = 0$ but leave θ to be arbitrary. Let $\Gamma = \{(x, y) \in \mathbb{R}^2 | x \ge 0\}$ and define a map $\Pi_{\mathcal{C}} : \Gamma \to \mathcal{C}$ such that $\Pi_{\mathcal{C}}(x, y) = [x, y]$. Note that the restriction of $\Pi_{\mathcal{C}}$ to the interior of Γ is a covering map for $\mathcal{C} - \{v\}$. Likewise define map $\Pi_{\mathcal{P}} : \Gamma \to \mathbb{R}^2 - \{\mathbf{0}\}$ such that $\Pi_{\mathcal{C}}(x, y) = (x \cos y, x \sin y)$. Then the restriction of $\Pi_{\mathcal{P}}$ to the interior of Γ is a covering map for $\mathbb{R}^2 - \{\mathbf{0}\}$.

Consider three distinct points $a_1, a_2, a_3 \in \mathcal{C} - \{v\}$. Let $\mathcal{T}^* \subset \mathcal{C} - \{v\}$ be an arbitrary tree connecting $\{a_1, a_2, a_3\}$, not necessarily consisting of geodesics. We desire to find the relatively minimal tree in the Steiner topology class $[\mathcal{T}^*]$, which we will denote as \mathcal{T}^*_{\min} . Since $\Pi_{\mathcal{C}}$ is a covering map, we can lift \mathcal{T}^* into Γ via $\Pi_{\mathcal{C}}$ and then project onto \mathbb{R}^2 via $\Pi_{\mathcal{P}}$. Let A_i be the image of a_i in \mathbb{R}^2 . Let $\hat{\mathcal{T}}$ be the solution to the planar Steiner problem on $\{A_1, A_2, A_3\}$ in \mathbb{R}^2 . Now since $\Pi_{\mathcal{P}}$ is a covering map, we can lift $\hat{\mathcal{T}}$ into Γ via $\Pi_{\mathcal{P}}$ and then project onto \mathcal{C} via $\Pi_{\mathcal{C}}$. Care is taken to that the image of A_i under these maps is again a_i . Let \mathcal{T} be the image of $\hat{\mathcal{T}}$. If $\mathcal{T} \in [\mathcal{T}^*]$, then $\mathcal{T}^*_{\min} = \mathcal{T}$. If not, then \mathcal{T}^*_{\min} will contain v. In the case of the narrow cone, \mathcal{T}^*_{\min} cannot be the *g*-minimal tree and hence the g-minimal tree can not be in the class $[\mathcal{T}^*]$. In the case of the wide cone, this case is momentarily disregarded, but then will be reconsidered in an additional step. The additional step is to add the v to the set of terminal points and consider the possible solutions to the resulting four point problem with a degeneracy at v. The compositions of maps above are easily continuously extended by identifying $v \in \mathcal{C}$ with $\mathbf{0} \in \mathbb{R}^2$. The relatively minimal tree amongst the minimals over all indicated Steiner topologies is the desired q-minimal tree.

The *n*-point problem now generalizes easily. Let $a_1, a_2, \ldots, a_n \in \mathcal{C} - \{v\}$ and $\mathcal{T}^* \subset \mathcal{C} - \{v\}$ be an arbitrary tree connecting $\{a_1, a_2, \ldots, a_n\}$. We desire to find \mathcal{T}^*_{\min} , the relatively relatively minimal tree in the Steiner topology class $[\mathcal{T}^*]$. Lift \mathcal{T}^* into Γ via $\Pi_{\mathcal{C}}$ and then project onto \mathbb{R}^2 via $\Pi_{\mathcal{P}}$. Let A_i be the image of a_i in \mathbb{R}^2 . Let $\hat{\mathcal{T}}$ be the solution to the planar Steiner problem on $\{A_1, A_2, \ldots, A_n\}$ in \mathbb{R}^2 . Next lift $\hat{\mathcal{T}}$ into Γ via $\Pi_{\mathcal{P}}$ and then project onto \mathcal{C} via $\Pi_{\mathcal{C}}$. Care is taken

so that the image of A_i under these maps is again a_i . Let \mathcal{T} be the image of $\hat{\mathcal{T}}$. If $\mathcal{T} \in [\mathcal{T}^*]$, then $\mathcal{T}^*_{\min} = \mathcal{T}$. If not, then \mathcal{T}^*_{\min} will contain v and in both cases of the narrow and wide cones, this tree is disregarded from consideration. An additional step in the case of the wide cone is that v is added to the set of possible terminal points. However, v is really a super terminal point in the sense that it is allowed to have degree ranging from 2 through n. (In the case of the planar Steiner problem, the maximum degree of a terminal is 3.) The relatively minimal tree amongst the minimals over all indicated Steiner topologies is the desired g-minimal tree.

This discussion is only intended to be a outline of the strategy for the general n-point problem. Details for the general n-point problem such as characterizing all possible Steiner topologies and pruning strategies to reduce the number of cases considered are beyond the scope of this paper, but may be the subject of future work.

7. Applications

It is a well known fact that any piecewise continuous surface may be approximated arbitrarily well by a piecewise linear surface. A piecewise linear surface is a surface composed of triangular cells, which meet only at edges or at vertices. On such a surface, a vertex may be represented by of either a narrow or a wide cone structure. The results in this paper may be useful in further research in solving the Steiner problem on on any piecewise linear surface and hence any piecewise continuous surface by approximation.

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