

SHORTEST PATH BETWEEN TWO POINTS ON THE REGULAR TETRAHEDRON

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ABSTRACT. In this paper, we show how to find the shortest path between two points on a regular tetrahedron. We describe a general solution to find the shortest path between n points that lie on the same face. In the case that two points lie on distinct faces, we describe a tiling of the plane and a cutting strategy that leads to finding the shortest path on the tetrahedron.

1. INTRODUCTION

In this paper, we will show how to find the shortest distance between two points on the regular tetrahedron. This problem is in a class of problems of shortest path networks on non-planar surfaces. A shortest path network is the union of paths connecting n points so that the total distance is minimized. The problem of finding the shortest path between two points on a tetrahedron is non-trivial because, in general, there are infinitely many straight line segments on the regular tetrahedron connecting two points.

We devise a cutting strategy to find the shortest distance between two points on the regular tetrahedron. This result is essential to solving shortest length network problems for n points, with $n > 2$.

2. OVERVIEW

Two points on a tetrahedron can share a face or lie on distinct faces. We first characterize the solution to a more general problem which is to find the shortest path network connecting n points that all lie on the same face. In particular, we prove a more general result which shows that the shortest path network containing n points on the same face of the tetrahedron is contained in that face. In the case that two points are on distinct faces, we describe a tiling and a cutting strategy to find the shortest path.

Advised by Dr. Denise Halverson at a BYU REU funded by the NSF.

3. POINTS ON A COMMON FACE

Theorem 1. *For n coplanar points in \mathbb{R}^k , the shortest path network connecting these points is contained in the convex hull of the points.*

Proof. Let $\{A_1, \dots, A_n\}$ be points contained in the plane \mathcal{Q} in \mathbb{R}^k . Let $\Pi : \mathbb{R}^k \rightarrow \mathbb{R}^2 \times \{\vec{0}\}$ be the standard projection map so that $\Pi(x, y, \vec{z}) = (x, y, \vec{0})$. Without loss of generality assume \mathcal{Q} is contained in $\mathbb{R}^2 \times \{\vec{0}\}$. Then Π is an isometry on \mathcal{Q} . Also, for any path network α in \mathbb{R}^k , $\ell(\Pi(\alpha)) \leq \ell(\alpha)$. Let ρ be the shortest path network connecting $\{A_1, \dots, A_n\}$ in \mathcal{Q} . Let β be any other path network connecting $\{A_1, \dots, A_n\}$ in \mathbb{R}^k . Hence $\Pi(\beta)$ is a path network connecting $\{\Pi(A_1), \dots, \Pi(A_n)\}$. Then

$$\ell(\rho) \leq \ell(\Pi(\beta)) \leq \ell(\beta).$$

Thus ρ is the shortest path network in \mathbb{R}^k connecting $\{A_1, \dots, A_n\}$. From classical results, we know that the shortest path network connecting n points in the plane is contained in the convex hull of the points. Therefore, ρ is contained in the convex hull of $\{A_1, \dots, A_n\}$. \square

Corollary 2. *If $\{A_1, \dots, A_n\}$ are points on the same face of a regular tetrahedron, then the shortest path network connecting the points is contained in that face.*

Corollary 3. *If p and q are vertices on T , then the shortest path connecting p and q is the edge, \overline{pq} .*

4. POINTS ON DISTINCT FACES

We will now consider the case where both points are on distinct faces. In order to efficiently identify all of the possible paths, we will tile the plane with the faces of the tetrahedron.

4.1. Tiling the Plane. Consider a regular tetrahedron with faces f_1, f_2, f_3 , and f_4 . A tile that will be constructed in the plane corresponding to f_i on T , will be denoted as F_i , but in our figures labeled as i . Cut along the edges and lay the faces out in the plane as in Figure 1.

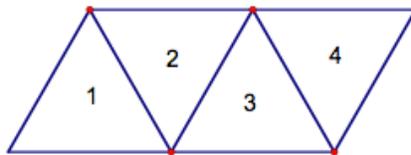


FIGURE 1

The tiling of the plane is generated by rotating copies of the faces 180° about their vertices. Note that one point on T corresponds to infinitely many points in the plane. For a point a on T we will denote a point corresponding to a in the tiled plane as A . This is illustrated in Figure 2. The projection map that rewraps the tiled plane around the tetrahedron matching each copy of F_i to f_i will be denoted as Ψ .

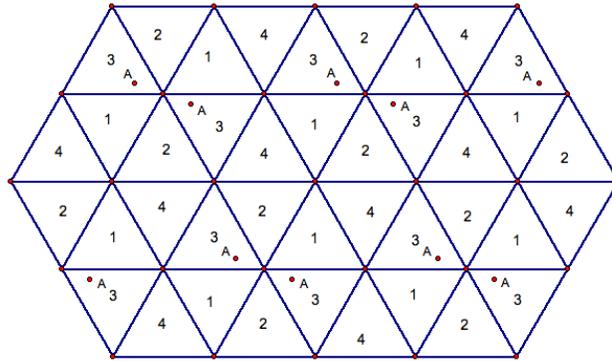


FIGURE 2

Definition 4. Let T be a regular tetrahedron and Ψ be the projection map that rewraps the tiled plane around the tetrahedron matching each copy of F_i to f_i . Let p be a point on T . Then a *lift* of p is a point P so that $\Psi(P) = p$.

Definition 5. Let T be a regular tetrahedron and Ψ be the projection map described in Definition 4. Let $\alpha : [0, 1] \rightarrow T$ be a path. Then a *lift* of α is a path $\alpha^* : [0, 1] \rightarrow \mathbb{R}^2$ so that the following diagram commutes:

$$\begin{array}{ccc}
 & \mathbb{R}^2 & \\
 \alpha^* \nearrow & & \searrow \Psi \\
 [0, 1] & \xrightarrow{\alpha} & T
 \end{array}$$

Remark 6. If a path does not pass through a vertex then its lift is unique. Note that the lift of a path would branch into two directions at any vertex and would not be unique. The argument showing uniqueness is similar to the proof of the path lifting theorem for covering spaces (See [5]).

We will show that it is not necessary to consider infinitely many paths in order to connect two points. We will describe a region, F -Star, that contains the lift of the shortest path between two points.

4.2. **F -Star.** Consider the tiling described in Section 4.1. For a specified point P in F_1 in the plane, $F_3[P]$ -Star is the region shown in Figure 3 which contains P in a hexagon that is bounded by copies of F_3 . In the given figure, F_3 -Star is outlined in bold. Note that $F_i[P]$ -Star is a different region for $i = 2, 3, 4$.

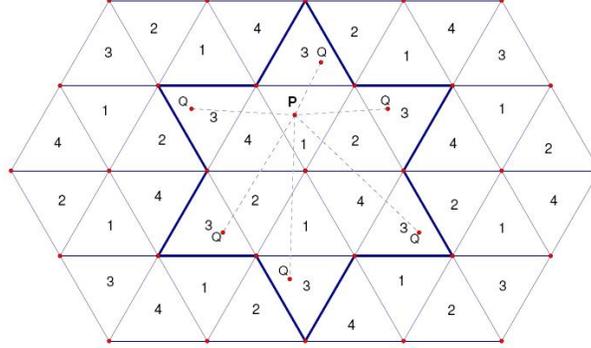


FIGURE 3

Theorem 7. *If p and q are on T , where p is in f_i , p is not a vertex of T , α is the shortest path from p to q on T , and P is a lift of p in \mathbb{R}^2 as described in Definition 4, then the lift of α based at P is contained in $F_i[P]$ -Star.*

Proof. Assume we have the labeling of the faces of T as described previously in Section 4.1. Without loss of generality, let p be on f_1 . Let α^* denote the lift of α .

Case 1: Suppose q is also on f_1 . Then by Corollary 2, α is contained on f_1 . By Definition 5, the lift of f_1 , which contains P , is contained in $F_1[P]$ -Star. Hence the lift of α based at P is in $F_1[P]$ -Star.

Case 2: Suppose q is not on f_1 . Without loss of generality, let q be on f_3 . Consider $F_3[P]$ -Star. Note that P is bounded by six copies of F_3 . We will prove this case by proving the contrapositive of the statement of the theorem. In particular, we will prove that if α^* based at P is not contained in $F_3[P]$ -Star, then α is not the shortest path from p to q on T .

Case a: Suppose α passes through a vertex. From [4] we know that the shortest path between two points on a cone never passes through the cone point. This can be generalized to say that the shortest path between two points on the regular tetrahedron never passes through the vertex. Thus, α is not the shortest path.

Case b: Suppose α does not pass through a vertex. Let Q be the endpoint of α^* and suppose α^* leaves $F_3[P]$ -Star. See Figure 4. Since α^* goes outside of $F_3[P]$ -Star, then it meets a copy of F_3 in a path from points A to B where A and B are in the boundary of the copy of F_3 . Note Q is outside of $F_3[P]$ -Star. Let a be the point on T corresponding to A . Let β be the sub-path from a to q . Then the lift of β must be a sub-path of α^* which is not entirely contained in a copy of F_3 . According to Corollary 2, α is not the shortest path.

□

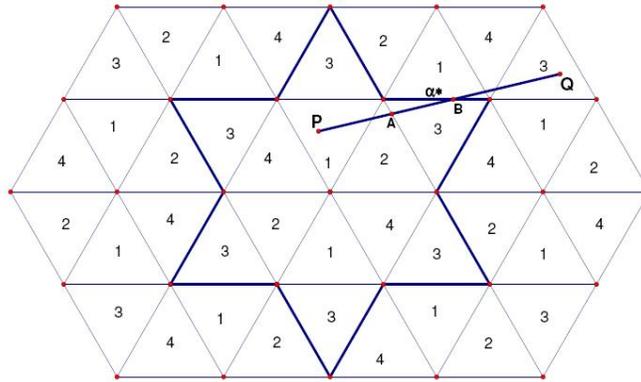


FIGURE 4

We will now show that it is not necessary to consider all paths within the F_i -Star region to connect two points. Here we describe a cutting strategy that will give three contending shortest paths to connect two points on the regular tetrahedron.

4.3. Cutting Strategy. Consider a regular tetrahedron and two points p, q on distinct faces. Cut from q to the three vertices on the same face as q . Label the vertex v that is not on the same face as q . Choose a point w such that $m\angle wvp = 90^\circ$. Cut along \vec{vw} . Lay the tetrahedron flat in the tiling so that the cut region, τ , contains P , a lift of p . Shown in Figure 5 is the case where q is on f_3 and p is on f_1 . Note that τ is contained in $F_3[P]$ -Star. There is another copy of τ also contained in $F_3[P]$ -Star, the rotation of τ by 180° , call this τ' . By Theorem 7, the lift of the shortest path is contained in $F_3[P]$ -Star. Indeed it is

contained in $\tau \cup \tau'$. (Any path connecting p to q can be reflected into $\tau \cup \tau'$ and then shortened.)

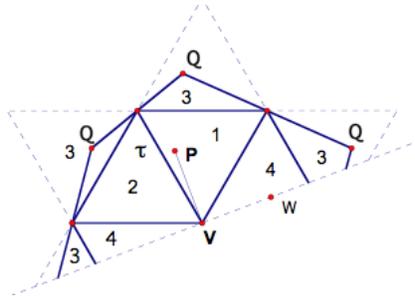


FIGURE 5

We will now show that the lift of α based at P is contained in τ .

Theorem 8. *Let p and q be on distinct faces of T , and let α be the shortest path between p and q . Let P be a point in the tiled plane corresponding to p . Suppose τ is the region described in the cutting strategy. Then the lift of α based at P , α^* , is contained in τ .*

Proof. Without loss of generality, suppose that p is on f_1 and q is on f_3 . Let p, q, P, τ, α , and α^* be defined as in the statement of the theorem. Let V be the center vertex of $F_3[P]$ -Star. Let m be the line perpendicular to \overline{PV} through V . From Theorem 7 we know that α^* is contained in $F_i[P]$ -Star. Let Q_1, Q_2, Q_3 be the points in τ corresponding to q and let Q'_1, Q'_2, Q'_3 be the points in τ' corresponding to q as in Figure 6. (In the case that there are four points in τ corresponding to q , two are common to both τ and τ' . We can arbitrarily place one of these points in the group Q_1, Q_2, Q_3 and the other point in Q'_1, Q'_2, Q'_3 .) We will now show that $PQ_i \leq PQ'_i$ and $PQ_i < PQ'_i$ if $Q_i \notin m$. It will follow that α^* is contained in τ .

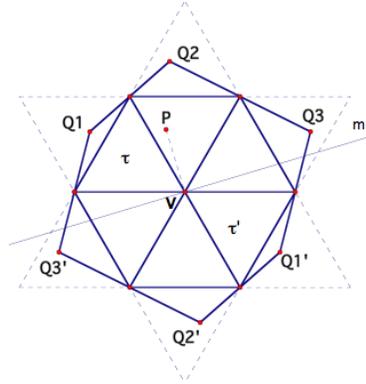


FIGURE 6

- Case 1: Let $i = 1$. If $Q_1 \in m$, then $PQ_1 = PQ'_1$. Otherwise, construct $\overline{Q_1Q'_1}$ and note that the midpoint is V . Construct the perpendicular bisector, k to $\overline{Q_1Q'_1}$. Let W be a point on m that is on the Q_3 side of V and let A be a point on k on the Q_3 side of V . Observe that P and Q_1 are on the same side of m . See Figure 7. It follows from the cutting strategy that $m\angle PVQ_1 \leq 90^\circ$. Also $m\angle PVQ_1 = m\angle AVW$. But $m\angle PVW = 90^\circ$ and so k only separates P and Q_1 if $m\angle AVW > 90^\circ$ which is not the case. Therefore, P and Q_1 are on the same side of k . Hence, $PQ_1 < PQ'_1$.
- Case 2: Let $i = 2$. Construct $\overline{Q_2Q'_2}$ and note that the midpoint is V . Construct the perpendicular bisector, k_2 to $\overline{Q_2Q'_2}$. Since P is on the Q_2 side of k_2 , $PQ_2 < PQ'_2$.
- Case 3: Let $i = 3$. This is the same argument as in Case 1.

□

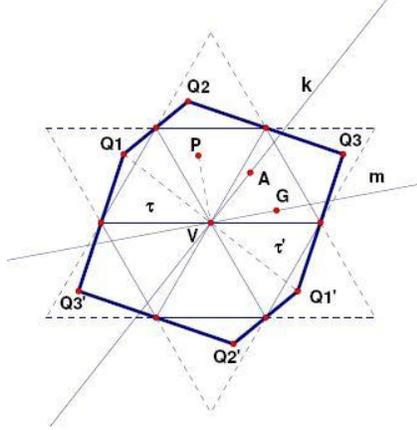


FIGURE 7

The following Lemma will help distinguish which of the three remaining options realizes the shortest segment connecting p and q .

Lemma 9. *The perpendicular bisectors of $\overline{Q_1Q_2}$, $\overline{Q_2Q_3}$, and $\overline{Q_1Q_3}$ meet at a common point.*

Proof. Let n_1 be the perpendicular bisector of $\overline{Q_1Q_2}$, n_2 be the perpendicular bisector of $\overline{Q_2Q_3}$, and n_3 be the perpendicular bisector of $\overline{Q_1Q_3}$. Let J be the point of intersection of n_1 and n_2 . Since n_1 is the perpendicular bisector of $\overline{Q_1Q_2}$ and J is on n_1 , $Q_1J = Q_2J$. Since n_2 is the perpendicular bisector of $\overline{Q_2Q_3}$ and J is on n_2 , $Q_2J = Q_3J$. Then $Q_1J = Q_3J$. Thus J is on the perpendicular bisector of $\overline{Q_1Q_3}$. Thus the perpendicular bisectors of $\overline{Q_1Q_2}$, $\overline{Q_2Q_3}$, and $\overline{Q_1Q_3}$ meet at a common point, J . \square

In order to complete the proof of Theorem 10, we introduce necessary points: Let M_1 be the midpoint of $\overline{Q_1Q_2}$, M_2 be the midpoint of $\overline{Q_2Q_3}$, and M_3 be the midpoint of $\overline{Q_1Q_3}$. Let N_3 be a point on n_3 so that N_3 lies in the exterior of $\angle M_1PM_2$. Let τ_1, τ_2, τ_3 be regions of τ where τ_1 is the region cut by $\angle M_1JN_3$, τ_2 is the region cut by $\angle M_1JM_2$, and τ_3 is the region cut by $\angle M_2JN_3$. (See Figure 8.)

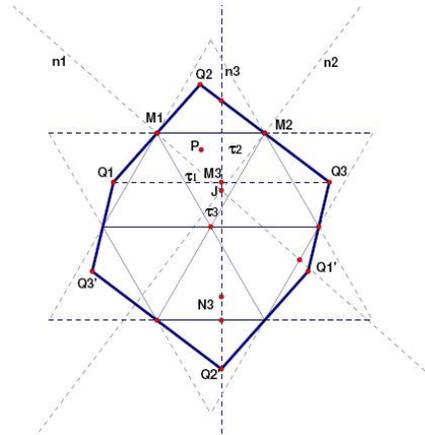


FIGURE 8

There are three possible shortest paths, $\overline{PQ_1}$, $\overline{PQ_2}$, and $\overline{PQ_3}$. The location of P and its relationship to the perpendicular bisectors n_1 , n_2 and n_3 determine the shortest path. The proof follows.

Theorem 10. *If P is in τ_i , then $PQ_i \leq PQ_j$ for $j \neq i$.*

Proof. If P is in τ_1 then its on the Q_1 side of n_1 and n_2 , and $\overline{PQ_1}$ is the shortest path. If P is in τ_2 then it is on the Q_2 side of n_1 and n_2 , and $\overline{PQ_2}$ is the shortest path. If P is in τ_3 then it is on the Q_3 side of n_1 and n_2 , and $\overline{PQ_3}$ is the shortest path. Thus $PQ_i \leq PQ_j$ for $j \neq i$. \square

5. APPLICATIONS

Gilbert and Pollack define a Steiner minimal as the tree which interconnects any number of given points in the plane using the shortest possible total length. In this paper, we have solved the 2-point problem on a regular tetrahedron.

Melzak's algorithm gives a solution to the n point problem in the plane. In the plane, the 3-point problem breaks down into 2-point problems, and the n point problem breaks down into a series of 3-point problems [3]. This paper gives an essential step to solving the n -point problem in general on the regular tetrahedron.

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