# THREE-POINT STEINER PROBLEM ON THE FLAT TORUS

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ABSTRACT. We will consider the Steiner Problem on a flat rectangular torus. This paper will focus on the three-point case.

## 1. INTRODUCTION

The Steiner problem, named after Jakob Steiner (1796-1863), is the problem of connecting a given (finite) set of points with the least-length path network. There is an algorithm, named after Zdzisław Melzak, that will yield the minimizer for any given finite set of points [2]. It has been shown that the minimizer must consist of line segments, each of which must have endpoints in the given set of fixed points, or in an additional set of Steiner points. With a set of n fixed points it has been shown that there can be no more than n - 2 Steiner points. At each Steiner point exactly 3 edges connect, forming angles of  $120^{\circ}$  [4], [5].

This problem has also been considered in other surfaces, such as the hyperbolic plane, which the author has worked on previously [1]. It has been shown that in surfaces of constant nonpositive curvature the same results hold—that a minimizer must consist of geodesics and when these geodesics meet at a Steiner point they need to form 120° angles.

This paper will consider the Steiner problem on the flat torus, which can be thought of as the space obtained from the rectangle  $R = [0, a] \times [0, b]$  by identifying opposing edges.

## 2. Definitions and Notation

There are several terms we will be using in this paper that we define here for the reference of the reader.

**Definition 2.1.** If  $f: X \to Y$  is a continuous surjective map then an open set  $U \subset Y$  is said to be *evenly covered* by f if  $f^{-1}(U)$  is a collection of disjoint open sets, each of which is homeomorphic to U. f is a *covering map* if for every point  $y \in Y$  there is a neighborhood of y that is evenly covered by f. If f is a covering map then we say that X is a *covering space* for Y. [3, p. 336]

Here we give a formal definition for the flat torus. Given l, w > 0, let  $\mathbb{Z}^2$  act on  $\mathbb{R}^2$  by  $(1,0) \cdot (x,y) = (x+l,y)$  and  $(0,1) \cdot (x,y) = (x,y+w)$ . Then we define  $T(l,w) = \mathbb{R}^2/\mathbb{Z}^2$  to be the orbit space under this action. In this paper T will be understood to mean T(l,w). Note that the map  $p: \mathbb{R}^2 \to T$  is a covering map.

**Definition 2.2.** Suppose  $f : X \to Y$  is a map. If  $g : Z \to Y$  is a continuous function, a *lifting* of g is a map  $\tilde{g} : Z \to X$  such that  $f \circ \tilde{g} = g$ . [3, p. 342]

Liftings are most useful when f is a covering map. We will use the following fact about liftings, but the proof will be omitted since it is beyond the scope of the paper.

**Lemma 2.1.** Let  $p : E \to B$  be a covering map, let  $p(e_0) = b_0$ . Any path  $f : [0,1] \to B$  beginning at  $b_0$  has a unique lifting to a path  $\tilde{f} : [0,1] \to E$  beginning at  $e_0$ . [3, p. 342]

This result will be important when we consider path networks on the torus, since we will lift them to the covering space  $\mathbb{R}^2$ . We say that R is a fundamental domain if it is a lifting of the identity  $i: T \to T$ , that is if f(R) = T and  $f|_R$  is a homeomorphism. A fundamental domain may not be rectangular; however, in this paper we will only discuss rectangular fundamental domains, since they are very simple and make calculations easy. We will also assume that for any such fundamental domain, one edge is parallel to the x-axis and the other parallel to the y-axis.

The following terms relate specifically to the Steiner problem. Let A be a set of n points on a surface. Then a *tree* is a path network connecting the points of A that contains no loops. The *minimal spanning tree* on A is the tree of minimal length where only paths with endpoints in A are allowed, (e.g., for three points, the minimal spanning tree would be tree connecting the two shorter sides of the triangle).

A Steiner tree is a locally minimal tree—one that consists of geodesics which are allowed to have endpoints not in A, but all such endpoints meet exactly three paths and the three angles formed are exactly  $120^{\circ}$ . These points are called Steiner points and it has been shown that there can be no more than n-2. A full Steiner tree is a Steiner tree with exactly n-2 Steiner points and a degenerate Steiner tree is a Steiner tree that is not full. The Steiner minimal tree is the minimal path network over all possibilities. In the Euclidean plane, it has been shown that when a full Steiner tree exists it is the Steiner minimal tree.

If  $\triangle$  is a triangle then we define  $\text{SMT}(\triangle)$  to be the Steiner minimal tree on the vertices of  $\triangle$  and  $\text{MST}(\triangle)$  to be the union of the two shorter sides of  $\triangle$ , which is the minimal spanning tree. When  $\triangle$  is understood, we will just use SMT and MST.

The following, while valid for the Steiner problem on any surface of zero curvature, will be particularly useful in our arguments about the torus. If A and B are two points we define the *lune* about them as follows: construct both equilateral triangles on  $\overline{AB}$  and circumscribe circles about them. The short arc of each circle with endpoints A and B will form the boundary of the lune. It has the special property that for any point C on that boundary  $\angle ACB = 120^{\circ}$  and for any point C in the interior,  $\angle ACB > 120^{\circ}$ . Now we draw the rays tangent to these circles at A and B and they have the property that for any point C on one of these lines,  $\angle BAC = 120^{\circ}$  or  $\angle ABC = 120^{\circ}$ , depending on the ray chosen. The *lune* will be the arcs described, together with the interior. The region bounded by the lune and its tangent rays will be the *Steiner region*, since for all points C in this region there is a full Steiner tree for  $\triangle ABC$ . The region bounded only by the tangent rays will be called the *degenerate region* since for any point C in it, the minimal tree for  $\triangle ABC$  is degenerate. We note that the lune is also a degenerate region, but it is of a different nature.





Another object that will be useful in this discussion is the notion of a curve based on two points A, B with the property that for any point C on that curve, SMT(ABC) is constant. We will call this curve, together with its bounded interior, a Steiner neighborhood. Its construction is fairly complicated, and counterintuitively we first define a large neighborhood and then a small one. Let A and Bbe points in the Euclidean plane. Then, for a given  $r \geq \frac{1}{2\sqrt{3}}AB$  we define the Steiner neighborhood of radius r about  $\overline{AB}$  as follows: Construct the standard Euclidean neighborhoods N(A,r) and N(B,r). Then define  $\mathcal{N}_1 = \partial N(A,r)$  and  $\mathcal{N}_2 = \partial N(B, r)$ . Construct the equilateral points  $E_1$  and  $E_2$  of A and B. Draw circles  $\mathscr{C}_1$  and  $\mathscr{C}_2$  centered at  $E_1$  and  $E_2$ , respectively, of radius AB + r. We note that each of  $\mathscr{C}_1$  and  $\mathscr{C}_2$  intersects each of  $\mathscr{N}_1$  and  $\mathscr{N}_2$  at exactly one point. Let  $p_1 = \mathscr{C}_1 \cap \mathscr{N}_1, p_2 = \mathscr{C}_2 \cap \mathscr{N}_1, p_3 = \mathscr{C}_2 \cap \mathscr{N}_2, and p_4 = \mathscr{C}_1 \cap \mathscr{N}_2.$  Then we construct the boundary of the Steiner neighborhood by taking the arc of  $\mathcal{N}_1$  from  $p_1$  to  $p_2$ , in the counterclockwise direction, together with the arc of  $\mathscr{C}_2$  from  $p_2$  to  $p_3$  and the arc of  $\mathcal{N}_2$  from  $p_3$  to  $p_4$ , and finally the arc of  $\mathcal{C}_1$  from  $p_4$  to  $p_1$ . We note that this curve has the desired property.

Now, the reason we left out Steiner neighborhoods of radius  $0 < r < \frac{1}{2\sqrt{3}}AB$ is that every tree in the lune is degenerate, so any Steiner neighborhood whose boundary intersects the lune must be modified. We note that for any point C in the lune, SMT(ABC) is the union of the segments  $\overline{AC}$  and  $\overline{BC}$ . Therefore, the curve along which any point C yields a constant value for AC + BC is an ellipse with A and B as the foci. Therefore, for  $r < \frac{1}{2\sqrt{3}}AB$ , let  $\mathscr{E}$  be the ellipse with foci A and B so that for any point  $C \in \mathscr{E}$ , AC + BC = AB + r, and we construct  $\mathcal{N}_1, \mathcal{N}_2, \mathcal{C}_1$ , and  $\mathcal{C}_2$  as above. We note that  $\mathscr{E}$  intersects  $\mathcal{C}_1$  in two points along the arc from  $p_4$  to  $p_1$ , call these points  $p_5$  and  $p_6$ . Similarly,  $\mathscr{E}$  intersects  $\mathcal{C}_2$  in two points between  $p_2$  and  $p_3$ , so we call these points  $p_7$  and  $p_8$ . Then the boundary of this Steiner neighborhood is similar to that of the above, but replace the arc of  $\mathscr{C}_1$ from  $p_5$  to  $p_6$  by the arc of  $\mathscr{E}$  from  $p_5$  to  $p_6$ , and similarly the arc of  $\mathscr{C}_2$  from  $p_7$  to  $p_8$  by the arc of  $\mathscr{E}$  from  $p_7$  to  $p_8$ .



FIGURE 2. An example of a Steiner neighborhood around the points A, B.

#### 3. EXISTENCE AND NON-UNIQUENESS OF MINIMIZER

The Steiner problem on the Euclidean plane is well-understood and it is wellknown that there is a unique minimizer for any given finite set of points. Existence on the torus follows from existence in the Euclidean case. That is, supposing A is a set of points in T and  $\tilde{A}$  is a set of liftings of those points then there is a unique minimizer  $\gamma$  for the points in  $\tilde{A}$ . Let l be the length of  $\gamma$ . Then the minimizer for A has length at most l. Therefore, any closed ball of radius l centered at a point in A must contain the minimizer.

Here we will note an example that shows that on the flat torus, a minimizer might not be unique. Consider the points  $A = (0, 0), B = (0, \frac{1}{3}), C = (0, \frac{2}{3}) \in T(1, 1)$ . It is obvious that this has no full Steiner tree, since the angle at each vertex of  $\triangle ABC$ is 180°. We also note that AB = BC = AC, so any of the three minimal spanning trees gives a minimizer. (It is noted that this is impossible in the Euclidean plane, for the only case where AB = BC = AC is when  $\triangle ABC$  is equilateral, in which case there is a full Steiner tree that is the unique minimizer.)

The reason this problem is more difficult than the Steiner Problem in the Euclidean plane is that there are an infinite number of liftings of each point from the torus to the plane, and therefore an infinite number of combinations of points, some having full Steiner trees, and some being degenerate. Therefore, we consider the following results that limit the number of combinations we must consider.

Since there are already ways of solving this problem on the Euclidean plane, the Steiner problem on the torus is really just that of picking the best combination of liftings of the points a, b, c in the torus. Therefore, we give a definition for the notion of a "winner." In general, if the points A, B, and C are liftings of a, b, c such that  $SMT(ABC) \leq SMT(A'B'C')$  for all other liftings A', B', and C', then ABC is a winning combination and these points are called winners. If, for a fixed lifting A of a, the liftings B and C of b and c form a smaller SMT with A than any other combination of translates of B and C then they are said to be winners relative to A. Similarly, if for the fixed points A and B, the point C forms the shortest SMT among all of its translates then it is said to be the winner relative to A, B.

## 4. Box Theorem for the Three-Point Problem

Here we discuss the Steiner Problem with three fixed points in the torus.

**Theorem 4.1** ("Don't go outside the box" theorem). Let  $p : \mathbb{R}^2 \to T$  be the standard covering map of T. Also, let  $a, b, c \in T$  be three distinct points. Then there is a Steiner minimal tree on a, b, c with a lifting that is contained in a rectangular fundamental domain.

*Proof.* We prove this by contradiction. First we show the weaker result that every lifting must be in a closed fundamental domain. Suppose  $\gamma$  is a lifting that is not contained in any closed fundamental domain. Let  $A = (x_1, y_1), B = (x_2, y_2), C = (x_3, y_3)$  be the liftings of A, B, C so that  $A, B, C \in \gamma$ . Define

$$m_x = \min\{x_1, x_2, x_3\} \quad M_x = \max\{x_1, x_2, x_3\} m_y = \min\{y_1, y_2, y_3\} \quad M_y = \max\{y_1, y_2, y_3\}.$$

Since  $\gamma$  does not lie in one closed rectangular fundamental domain, it follows that either  $M_x - m_x > l$  or  $M_y - m_y > w$ . Without loss of generality, we assume the



FIGURE 3. A sketch of the proof of this theorem. Note that CB' < CB.

former case. We also assume that  $m_x = x_1$  and  $M_x = x_2$  (that A is the "leftmost" point and B is the "rightmost" point).

Now we define the point  $P = (x_4, y_4)$  as follows. If  $\gamma$  is a full Steiner tree, let P be the Steiner point of  $\gamma$ . Otherwise, let P = C. Now we see that  $x_2 - x_1 = (x_2 - x_4) + (x_4 - x_1) > l$ , therefore either  $x_4 - x_1 > \frac{l}{2}$  or  $x_2 - x_4 > \frac{l}{2}$ . Again, without loss of generality, we assume the latter case. Here we note that there is a lifting  $B' = (x_2 - l, y_2)$  of the point B and that  $|x_4 - (x_2 - l)| < |x_4 - x_2|$  since  $x_2 - x_4 > \frac{l}{2}$ . Therefore, d(P, B') < d(P, B), so the tree  $\gamma' = (\gamma - \overline{PB}) \cup \overline{PB'}$  is shorter than  $\gamma$ , contradicting the assumption that  $\gamma$  was minimal. Now we see that in the case that  $\gamma$  lies in one closed fundamental domain but not in the fundamental domain itself, d(P, B') = d(P, B), so  $\gamma$  and  $\gamma'$  have the same length  $\gamma'$  is a tree that has the desired properties.

The reason this theorem is so useful is that it vastly limits the number of cases that we need to consider. If we create a rectangle  $R = [x, x+2l) \times [y, y+2w)$ , which would contain four fundamental domains, the minimizer must have a lift into R. Furthermore, since any translate of a fundamental domain is also a fundamental domain, we may assume that one of the three points has a lifting that is exactly in the center of R, we really only need to check 16 cases. The other reason this result is useful is that it has a nice generalization to a higher number of points that will be shown later.

## 5. Elimination of Competitors Using the Steiner Ratio

Another result that will be even more useful is a simple application of the Steiner ratio. The Steiner ratio is defined to be  $r = \inf \frac{\text{Length}(\text{SMT})}{\text{Length}(\text{MST})}$ , where the infimum is taken over all finite sets of at least two points. It is conjectured that this value is  $\frac{\sqrt{3}}{2}$ . Indeed, that conjecture has been proven in the case where only three-point sets are considered.<sup>1</sup> We will use this result to further limit the number of points we need to consider.

Let  $a, b, c \in T$  and assume that O = (0,0) is a lift of the point a. Let  $B_1$ ,  $B_2, B_3, B_4, C_1, C_2, C_3, C_4$  be the liftings of b and c that lie in the rectangle  $[-l, l) \times [-w, w)$  so that  $B_i$  and  $C_i$  lie in the  $i^{th}$  quadrant. Of all 16 possible combinations of  $O, B_i, C_j$ , let  $OB_m C_n$  be the triplet that has the shortest minimal spanning tree. Then this combination is said to yield the *minimal MST*. Denote the length of this tree by  $\rho$ . Now it follows from the Steiner ratio that any combination of points  $B_i, C_j$  that, together with O, has a minimal spanning tree of length  $\frac{2}{\sqrt{3}}\rho$ ,

<sup>&</sup>lt;sup>1</sup>insert citation here

that combination cannot generate the minimizing tree. It is difficult in general to say exactly how many combinations this algorithm will rule out, but we will consider an interesting result.

Assume the fundamental domain is a square of unit length. Let  $B_1 = (0, \frac{1}{2})$ ,  $C_1 = (\frac{1}{2}, \frac{1}{4})$ . We claim that of all possible points  $\tilde{B}, \tilde{C} \in [0, 1) \times [0, 1)$  these two, together with the origin, have the largest possible minimal spanning tree in T(1, 1). The algorithm we have just described reduces the number of possibilities from 16 to 9. The main reason this algorithm is not better for this configuration is that many of the combinations generate minimal spanning trees with equal lengths. Now that we have seen a case where this algorithm does not do as well as we could hope, we state a lemma that will let us know when we have a winner.

**Lemma 5.1.** Let  $a, b, c \in T$ , and let A, B, C be lifts of those points repsectively. If  $AB, AC, BC < \frac{\sqrt{3}}{4}m$  (where  $m = \min\{l, w\}$ ) then SMT(ABC) is a lifting of the minimizer for the points a, b, c in the torus.

*Proof.* First we assume that A = (0,0). We note that Length  $\left(\operatorname{MST}(ABC)\right) \leq AB + AC < \frac{\sqrt{3}}{2}m$  and that for any translate B' of B or C' of C, we have  $AB', AC' \geq (1 - \frac{\sqrt{3}}{4})m$ . Since  $1 - \frac{\sqrt{3}}{4} > \frac{1}{2}$ , we see that  $\frac{2}{\sqrt{3}}$  Length  $\left(\operatorname{MST}(ABC)\right) < m < AB' + AC'$ . Therefore, any triangle  $\triangle AB'C'$  that has  $AB' \cup AC'$  as its minimal spanning tree cannot be the minimzer, by the Steiner ratio.

The cases we have left are when MST(AB'C') is either  $AB' \cup B'C'$  or  $AC' \cup B'C'$ . Without loss of generality, we may assume the former. Now we note that since  $BC < \frac{\sqrt{3}}{4}m$ , for any translate of B or C we have  $B'C' \ge BC$ . Therefore,

$$\operatorname{Length}\left(\operatorname{MST}(AB'C')\right) \geq AB' + BC > 1 - \frac{\sqrt{3}}{4} + BC$$

and

$$\operatorname{Length} \big(\operatorname{MST}(ABC)\big) < \frac{\sqrt{3}}{4} + BC,$$

 $\mathbf{SO}$ 

$$\frac{2}{\sqrt{3}}\operatorname{Length}\left(\operatorname{MST}(ABC)\right) < \frac{1}{2} + \frac{2}{\sqrt{3}}BC$$

and since  $BC < \frac{\sqrt{3}}{4}$ , we have

$$\begin{array}{rcl} \displaystyle \frac{2-\sqrt{3}}{\sqrt{3}}BC &< \displaystyle \frac{2-\sqrt{3}}{4} \\ \displaystyle (\frac{2}{\sqrt{3}}-1)BC &< \displaystyle \frac{1}{2}-\frac{\sqrt{3}}{4} \\ \displaystyle \frac{1}{2}+\frac{2}{\sqrt{3}}BC &< \displaystyle 1-\frac{\sqrt{3}}{4}+BC \\ \displaystyle \frac{2}{\sqrt{3}}l(\mathrm{MST}(ABC)) &< \displaystyle l\big(\mathrm{MST}(AB'C')\big) \end{array}$$

Therefore, by the Steiner ratio, it follows that AB'C' cannot yield the minimizer, so it must be the case that SMT(ABC) is the minimizer for A, B, C.

#### 6. Linking SMT and MST

Let A, B, and C be liftings of a, b, and c in T that yield the minimal MST. Although in many cases, this combination also yields the minimal SMT, and therefore the minimizer for the Steiner problem, it is not true in all cases. We have used Mathematica to find examples where this is not the case, for T(1, 1). Although we do not have a proof, we believe that there are such examples for any torus T(l, w).

One result we have to determine when the minimal MST coincides with the minimal SMT is in T(1,1) and in the case where the points A and B (labeled to be the vertices of the longest edge of the MST) lie on a vertical or horizontal line. Without loss of generality, we will assume that this is a horizontal line. Then given a point C defining a Steiner neibhorhood around A and B, we define r to be the radius of this neighborhood. Then the height of the Steiner neighborhood is  $2(1-\frac{\sqrt{3}}{2})AB + 2r$  and if this value is less than 1 then we are guaranteed that there are no translates of C in the Steiner neighborhood, therefore indicating that among all translates of C, that one is best for the chosen pair A, B. Solving for r we have that if  $r < \frac{1}{2} - (1 - \frac{\sqrt{3}}{2})AB$  then C will be the winner relative to A, B. Now we note, since A and B lie on the same horizontal line, that  $AB \leq \frac{2}{3}$  therefore,

$$\frac{1}{2} - (1 - \frac{\sqrt{3}}{2})AB \ge \frac{1}{2} - (1 - \frac{\sqrt{3}}{2})\frac{2}{3} > .41,$$

so if r < .41 then C is the winner relative to A, B. We note that this is the case for most possible points C.

Now we try a slightly more general result, also in T(1, 1), that allows for no restriction on the orientation of  $\overline{AB}$ . In this case we wish to calculate the diameter of the Steiner neighborhood, excluding the degenerate regions. We are not concerned with the degenerate regions because we have already assumed that there are no translates of C in those regions by choosing the minimal MST. We let s be the diameter of this region and use the law of cosines to see that  $s^2 = 4r^2 + (AB)^2 +$ 2r(AB). With the same reasoning as before, we wish to bound s above by 1, so we have s < 1, equivalently  $s^2 < 1$ , so solving for r we get

$$r < \frac{-2(AB) + \sqrt{16 - 12(AB)^2}}{8}$$

guarantees C to be a winner. This time we see that  $AB \leq \frac{\sqrt{2}}{2}$ , and therefore r < .21 would guarantee C to be a winner. This result is much weaker, but it is concievable that there are other arguments that may be found to help strengthen it.

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