THE LYNDON-HOCHSCHILD-SERRE SPECTRAL SEQUENCE FOR A PARABOLIC SUBGROUP OF $\mathrm{GL}_n(\mathbb{Z})$

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Abstract. Let Γ be a congruence subgroup of level N in $\mathrm{GL}_n(\mathbb{Z})$. Let P be a maximal $\mathbb{Q}$-parabolic subgroup of $\mathrm{GL}_n/\mathbb{Q}$, with unipotent radical U, and let $Q = (P \cap \Gamma)/(U \cap \Gamma)$. Let $p > \dim_{\mathbb{Q}}(U(\mathbb{Q})) + 1$ be a prime number that does not divide N. Let M be a (U, p)-admissible Γ-module. Consider the Lyndon-Hochschild-Serre spectral sequence arising from the exact sequence $1 \to U \cap \Gamma \to P \cap \Gamma \to Q \to 1$, which abuts to $H_*(P \cap \Gamma, M)$. We show that if M is a trivial $U \cap \Gamma$-module, then certain classes in the E^2 page survive to E^∞. We use this to obtain information about classes in $H_*(P \cap \Gamma, M)$ even if M is not a trivial $U \cap \Gamma$-module. This information will be used in future work to prove a Serre-type conjecture for sums of two irreducible Galois representations.

1. Introduction

Fix a prime number p and an algebraic closure $\mathbb{F}$ of the prime field of characteristic p. In this note we study the homology of maximal parabolic subgroups G of congruence subgroups of $\mathrm{GL}_n(\mathbb{Z})$ with coefficients in certain $\mathbb{F}$-vector spaces. This involves a Lyndon-Hochschild-Serre (from now on "LHS") spectral sequence that abuts to the homology of G.

We use bold letters to denote algebraic groups. If J is a group and V is a J-module, then V^J denotes the fixed points of V under J. If ϵ is a character of a group G, let $\mathbb{F}_\epsilon$ denote the one-dimensional space on which G acts via ϵ.

Definition 1.1. Let $A_1, \dots, A_k$ be positive integers with $A_1 + \cdots + A_k = n$. A parabolic subgroup of $\mathbf{GL}_n$ or of $\mathrm{GL}_n(\mathbb{Q})$ is called *standard* of type $(A_1, \dots, A_k)$ if it consists of lower block diagonal matrices with blocks of sizes $A_1, \dots, A_k$. If a parabolic subgroup is conjugate to a standard parabolic subgroup of type $(A_1, \dots, A_k)$, then we say it also has type $(A_1, \dots, A_k)$.

Every parabolic subgroup of $\mathbf{GL}_n$ or of $\mathrm{GL}_n(\mathbb{Q})$ is conjugate to a standard parabolic subgroup by matrix in $\mathrm{GL}_n(\mathbb{Z})$.

Definition 1.2. Let P be a parabolic subgroup of $\mathrm{GL}_n(\mathbb{Q})$ with unipotent radical U. Let Γ be a subgroup of $\mathrm{GL}_n(\mathbb{Z})$, and let $G = \Gamma \cap P$. A (U, p)-*admissible* G-module M is a G-module of the form $V \otimes \mathbb{F}_\epsilon$ where V is an irreducible module for $\mathbb{F}\mathbf{GL}_n(\mathbb{Z}/p)$ on which G acts via its reduction modulo p, and $\epsilon : G \to \mathbb{F}^\times$ is a character that is trivial on $G \cap U$.

A character ϵ as in this definition is called a nebentype character. For example, let $e : \mathbb{Z}/N \to \mathbb{F}^\times$ be a character. Let $\Gamma = \Gamma_0(N)$ be the subgroup of $\mathrm{GL}_n(\mathbb{Z})$

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whose first row (with the exception of the first entry) is congruent to 0 modulo N. For $\gamma \in \Gamma_0(N)$, let $\epsilon(\gamma) = e(\gamma_{11})$, and let $\mathbf{P}$ be any maximal $\mathbb{Q}$-parabolic subgroup. Let $\mathbf{P}_0$ be the standard parabolic subgroup conjugate to $\mathbf{P}$. Then ϵ restricted to $G = \Gamma_0(N) \cap \mathbf{P}$ is a nebentype character. If ϕ is an automorphism of $\mathrm{GL}_n(\mathbb{Z})$, and $\phi(\mathbf{P}) = \mathbf{P}_0$, then $\epsilon \circ \phi^{-1}$ is a nebentype character on $\phi(\Gamma_0(N)) \cap \mathbf{P}_0$.

Given a prime p, an n-tuple $(a_1, \dots, a_n)$ of integers is p-restricted if $0 \leq a_n < p-1$, and $0 \leq a_i - a_{i+1} < p$ for $1 \leq i < n$. Irreducible modules for $\mathbb{F}\mathbf{GL}_n(\mathbb{Z}/p)$ are classified by their highest weights, which are necessarily p-restricted. We use the notation $F(a_1, \dots, a_n)$ for the irreducible module with highest weight $(a_1, \dots, a_n)$. In this paper, we will assume throughout that $V = F(a_1, \dots, a_n)$.

Definition 1.3. Let N be a positive integer. A subgroup Γ of $\mathrm{GL}_n(\mathbb{Z})$ is *determined by congruence conditions* modulo N if it is the full preimage of a subgroup of $\mathrm{GL}_n(\mathbb{Z}/N)$ under the reduction modulo N map. Note that this implies that Γ contains the principal congruence subgroup modulo N.

Our main theorem has two parts.

Theorem 1.4. *Let Γ be a subgroup of $\mathrm{GL}_n(\mathbb{Z})$ determined by congruence conditions modulo an integer N, and let p be a prime that does not divide N. Let $\mathbf{P} = \mathbf{LU}$ be a maximal $\mathbb{Q}$-parabolic subgroup of $\mathbf{GL}_n$, where $\mathbf{U}$ is its unipotent radical and $\mathbf{L}$ is a Levi-factor. Let $G = \mathbf{P} \cap \Gamma$, $H = \mathbf{U} \cap \Gamma$, $Q = G/H$. Let M be a (U, p)-admissible G-module.*

(a) For any m, the natural map $H_m(G, M^H) \to H_m(G, M)$ is injective.

(b) If M' is any submodule of M, consider the LHS spectral sequence $E(M')$ with coefficients in M' for the exact sequence

$$1 \to H \to G \to Q \to 1.$$

Let d be the rank of the free abelian group H, and assume that $p > d + 1$. Suppose there is a nonzero $z \in E^2_{jd}(M^H) = H_j(Q, H_d(H, M^H))$ for some j. Then z survives to a nonzero element of $E^\infty_{jd}(M^H)$.

Remark 1.5. The exact sequence of groups mentioned in the theorem does not split in general, which increases the difficulty of the proof of the theorem. Also, it is very unlikely that the LHS spectral sequence in the theorem degenerates, even though if M is replaced by a $\mathbb{Q}$-vector space it is known that the resulting LHS spectral sequence does degenerate at E^2 (see [7, Theorem 2.7]).

In [1], we use Theorem 1.4 to study the following question. Let $G_\mathbb{Q}$ be the absolute Galois group of $\mathbb{Q}$. A *Galois representation* is a continuous homomorphism $\rho : G_\mathbb{Q} \to \mathrm{GL}_n(k)$ for some topological field k. We say that ρ is *odd* if ρ applied to complex conjugation has eigenvalues $\pm(1, -1, 1, -1, \dots)$. Given an odd Galois representation $\rho : G_\mathbb{Q} \to \mathrm{GL}_n(\mathbb{F})$, does there exist a level N and an irreducible $\mathbb{F}[\Gamma_0(N)]$-module M, and a Hecke eigenclass $z \in H_*(\Gamma_0(N), M)$ with ρ attached?

In [1], we show that the answer is "yes" if ρ has squarefree Serre conductor and is the direct sum of two irreducible representations, each of smaller dimension, and each attached to Hecke eigenclasses. That paper depends on the main results of this paper. We first find an element in $E^2_{jd}(M^H)$ that has ρ attached to it. We use (b) to show that there is an element in $H_{j+d}(G, M^H)$ that has ρ attached. We then use (a) to get an element in $H_{j+d}(G, M)$ with ρ attached.

Here is a sketch of the proof of Theorem 1.4. We consider a certain semigroup Σ of "semi-scalar" matrices. These are matrices in $\mathbf{P}(\overline{\mathbb{Q}})$ whose coefficients are algebraic integers prime to p and which are in the center of a Levi component of $\mathbf{P}$. They act on M through their reduction modulo a prime above p. We also need another semigroup of semi-scalar matrices:

$$\Sigma(\mathbb{Z}, N) = \{x \in \Sigma \cap M_n(\mathbb{Z}) \mid x \equiv I \mod N\}.$$

The semigroup Σ acts only on M, while $\Sigma(\mathbb{Z}, N)$ acts on both a resolution of H and on M.

For (a), in section 3 we consider a filtration of M by G-modules such that each quotient is a trivial H-module. The spectral sequence arising from this filtration has a semisimple action of Σ on it. Tracking the eigencharacters of Σ on the various terms of this spectral sequence provides a proof of the injection.

We prove (b) in section 4. Because d is the homological dimension of H, z vanishes under all the higher differentials. We show that none of the images of z in subsequent pages of the spectral sequence can be in the image of a higher differential, and that implies (b).

We are able to prove this for M^H-coefficients because we have good control on $H_j(G, M^H)$ as a Q-module and as a $\Sigma(\mathbb{Z}, N)$-module. (We do not have this control if we replace M^H with M.) This gives a semisimple action of $\Sigma(\mathbb{Z}, N)$ on all pages of the LHS spectral sequence, commuting with the differentials. We are able to separate the eigenalues of $\Sigma(\mathbb{Z}, N)$ on z from those of anything that could possibly map onto it under a higher differential.

All modules in this paper are right-modules, unless otherwise stated.

2. The LHS spectral sequence

Suppose we have an exact sequence of groups, with abelian kernel:

$$1 \to H \to G \to Q \to 1.$$

This gives an action of Q on H, whether or not the sequence is split, by setting $h \bullet q = g^{-1}hg$ for any lift g of q to G. The action does not depend on the lift. We call this the (natural) Q-action on H.

Fix a ring k. Let F be a resolution of k by free kG-modules (for example the standard resolution of G) and let Φ be a resolution of k by free kQ-modules (for example the standard resolution of Q). Let M be a kG-module. Form the double complex

$$C_{ij} = \Phi_i \otimes_Q (F_j \otimes_H M).$$

Recall that $F_j \otimes_H M$ is a Q-module under the diagonal action because $F_j \otimes_H M = (F_j \otimes_k M)_H$. Starting with this double complex, and taking the homology first in the j-direction and then in the i-direction gives rise to the LHS-spectral sequence:

$$E^2_{ij} = H_i(Q, H_j(H, M)) \implies H_{i+j}(G, M).$$

3. Injectivity

Theorem 3.1. *Let Γ be a subgroup of $\mathrm{GL}_n(\mathbb{Z})$ determined by congruence conditions modulo N. Assume that p is prime to N. Let $\mathbf{P} = \mathbf{LU}$ be a maximal $\mathbb{Q}$-parabolic subgroup of the algebraic $\mathbb{Q}$-group $\mathbf{GL}_n$. Let $G = \mathbf{P} \cap \Gamma$, $H = \mathbf{U} \cap \Gamma$, $Q = G/H$. Let M be a (U, p)-admissible module. Then the map induced by inclusion $\iota : M^H \to M$*

$$\iota_* : H_j(G, M^H) \to H_j(G, M)$$

is injective for any j.

The proof of this theorem will take up the rest of this section. Suppose that $\mathbf{P}$ has type (A, B). Conjugating everything by an element of $\mathrm{GL}_n(\mathbb{Z})$, we may assume that $\mathbf{P} = \mathbf{P}_0$. We will continue this assumption throughout the paper.

Throughout this section, we define Γ, G, H, and Q, as in Theorem 3.1.

Lemma 3.2. *(1) The inclusion $M^H \to M$ is G-equivariant.*

(2) If M' is a G-module on which H acts trivially, then $H_d(H, M')$ is naturally isomorphic as G-module to $\wedge^d H \otimes_{\mathbb{F}} M'$, where G acts on H via conjugation.

Proof. (1) This is clear since H is normal in G.

(2) The Pontryagin product is natural [4, pg. 122], so $H_d(H, M')$ is naturally isomorphic to $\wedge^d H \otimes_{\mathbb{F}} M'$. Since G acts on H by conjugation, and conjugation is an automorphism of H, this is an isomorphism of G-modules. □

We now introduce some notation for irreducible modules. Let $F(a_1, \ldots, a_n)$ denote the unique irreducible $\mathbb{F}\,\mathrm{GL}_n(\mathbb{F})$-module with highest weight $(a_1, \ldots, a_n)$. We also let $F(a_1, \ldots, a_A; b_1, \ldots, b_B)$ denote the irreducible $\mathbb{F}(\mathrm{GL}_A(\mathbb{F}) \times \mathrm{GL}_B(\mathbb{F}))$-module $F(a_1, \ldots, a_A) \otimes_{\mathbb{F}} F(b_1, \ldots, b_B)$. We use the same notation to denote the restriction of this module to any subgroup of $\mathrm{GL}_A(\mathbb{F}) \times \mathrm{GL}_B(\mathbb{F})$. In particular, $\mathbf{P}(\mathbb{Z})/\mathbf{U}(\mathbb{Z})$ is isomorphic to $\mathrm{GL}_A(\mathbb{Z}) \times \mathrm{GL}_B(\mathbb{Z})$, so that we can consider G/H modulo p to be a subgroup of $\mathrm{GL}_A(\mathbb{F}) \times \mathrm{GL}_B(\mathbb{F})$ and thus to act on $F(a_1, \ldots, a_A; b_1, \ldots, b_B)$.

Lemma 3.3. *There are natural isomorphisms of G-modules*

$$\begin{aligned} H_d(H, M) &\cong H_d(H, M^H) \\ &\cong \wedge^d H \otimes_k M^H \\ &\cong F(a_1 + B, \ldots, a_A + B; a_{A+1} - A, \ldots, a_n - A)_\epsilon. \end{aligned}$$

Proof. This follows immediately from [2, Theorem 9.1] and its proof. □

In order to show that $\iota_* : H_j(G, M^H) \to H_j(G, M)$ is injective, we create a filtration of the G-module M whose associated graded module is a trivial H-module. Examining the spectral sequence associated to this module, we find that $E^1_{0q} = H_q(G, M^H)$. Using the action of semiscalar matrices on this module, we show that E^1_{0q} is equal to E^∞_{0q}, which is the image of $H_q(G, M^H)$ in $H_q(G, M)$, so the map ι_* is injective.

Lemma 3.4. $M^H = M^{\mathbf{U}(\mathbb{F})}$.

Proof. Γ is defined by congruence conditions modulo N, so its reduction modulo p contains $\mathrm{SL}_n(\mathbb{Z}/p)$. Therefore, the reduction of $H = \Gamma \cap \mathbf{U}(\mathbb{Z})$ modulo p equals all of $\mathbf{U}(\mathbb{Z}/p)$. Thus $M^H = M^{\mathbf{U}(\mathbb{Z}/p)}$. It follows from [6, Corollary p. 51] (at the end of section 10 of chapter 5), that $M^{\mathbf{U}(\mathbb{Z}/p)} = M^{\mathbf{U}(\mathbb{F})}$. □

By the definition of M, $\mathbf{GL}_n(\mathbb{F})$ acts on M. Choose a prime $\mathfrak{p}$ over p in the ring of integers $\mathcal{O}$ of $\overline{\mathbb{Q}}$, and fix an isomorphism $\mathcal{O}/\mathfrak{p} \to \mathbb{F}$. Let $\mathcal{O}_\mathfrak{p}$ be the localization of $\mathcal{O}$ at $\mathfrak{p}$. We obtain an action of $\mathrm{GL}_n(\mathcal{O}_\mathfrak{p})$ on M by first reducing modulo $\mathfrak{p}$ and then applying the $\mathbf{GL}_n(\mathbb{F})$ action on M.

Definition 3.5. Define the group Σ of *semi-scalar matrices* in $\mathbf{GL}_n(\mathcal{O}_\mathfrak{p})$ by

$$\Sigma = \{s_\alpha = \mathrm{diag}(\alpha I_A, I_B) : \alpha \in \mathcal{O}_\mathfrak{p}^\times\}.$$

Let $\overline{\Sigma}$ be the reduction of Σ modulo $\mathfrak{p}$.

Lemma 3.6. *Let $s \in \Sigma$ and $x \in \mathbf{P}(\mathcal{O}_\mathfrak{p})$. Then $s^{-1}xs = hx$ for some $h \in \mathbf{U}(\mathcal{O}_\mathfrak{p})$.*

Proof.
$$\begin{bmatrix} \alpha^{-1}I_A & 0 \\ 0 & I_B \end{bmatrix} \begin{bmatrix} a & 0 \\ b & c \end{bmatrix} \begin{bmatrix} \alpha I_A & 0 \\ 0 & I_B \end{bmatrix} = \begin{bmatrix} I_A & 0 \\ u & I_B \end{bmatrix} \begin{bmatrix} a & 0 \\ b & c \end{bmatrix}$$
where $u = (b\alpha - b)a^{-1}$. Since a is invertible in $\mathrm{GL}_A(\mathcal{O}_\mathfrak{p})$, the lemma follows. □

Let $\overline{s_\alpha} \in \overline{\Sigma}$ be the reduction of s_α modulo $\mathfrak{p}$, so s_α acts on M via $\overline{s_\alpha}$. Since $\overline{\Sigma}$ consists of elements whose orders are prime to p, its action on M can be diagonalized. In fact, we can diagonalize M with respect to the whole diagonal torus in $\mathrm{GL}_n(\mathcal{O}_\mathfrak{p})$, and the eigencharacters that appear are the weights of the representation.

Lemma 3.7. *(1) M^H is an irreducible $\mathbf{L}(\mathbb{F})$-module isomorphic to*
$$F(a_1, \ldots, a_A; a_{A+1}, \ldots, a_n).$$
(2) The weights of the diagonal matrices on M/M^H are all of the form
$$(b_1, \ldots, b_n)$$
such that $0 \le b_1 + \cdots + b_A < a_1 + \cdots + a_A$.

Proof. We apply [6, pp. 51-2] to $M = F(a_1, \ldots, a_n)$. Assertion (1) is [6, Corollary, p. 51]. Let $\lambda = (a_1, \ldots, a_n)$. Assertion (2) follows from the statement in [6, pg. 50] which (rewritten in our notation and taking the corollary on page 51 of [6] into account) says that $M = M^H \oplus M^*$ where M^* is the sum of all the weight spaces with weights μ of the form
$$\mu = \lambda - \xi$$
where ξ is a weight that does not lie in $\mathbb{Z}\Theta^+$. Here Θ^+ denotes the set of positive roots of $\mathbf{L}$. For $(c_1, \ldots, c_n)$ to lie in $\mathbb{Z}\Theta^+$, it must be an integral linear combination of $e_i - e_j$, where i, j are either both in $\{1, \ldots, A\}$ or both in $\{A+1, \ldots, n\}$. Therefore, in the expression of ξ as a linear combination of the basis $\{e_k - e_{k+1}\}$, the basis element $e_A - e_{A+1}$ must appear with a nonzero coefficient.

Now we know (cf. [3, proof of Lemma 6.1 (2)]) that in fact the coefficients in ξ are all non-negative. So we are subtracting off $e_A - e_{A+1}$ a positive number of times to get μ. This gives the upper bound. The lower bound follows because any irreducible $\mathbb{F}[\mathbf{GL}_n(\mathbb{F})]$-module is isomorphic to a subquotient of a tensor product of fundamental irreducible representations of $\mathbf{GL}_n$. □

Corollary 3.8. *(1) For any $\alpha \in \mathcal{O}_\mathfrak{p}^\times$, the eigenvalues of s_α on M^H are all equal to*
$$\overline{\alpha}^{a_1 + \cdots + a_A}$$
while the eigenvalues of s_α on M/M^H are equal to
$$\overline{\alpha}^{b_1 + \cdots + b_A}$$
for various $b_1, \ldots, b_A$, all of which satisfy $0 \le b_1 + \cdots + b_A < a_1 + \cdots + a_A$.

(2) The eigencharacters of $\overline{\Sigma}$ on M^H are pairwise distinct from the eigencharacters of $\overline{\Sigma}$ on M/M^H.

Proof. (1) follows immediately from the preceding lemma.

(2) Suppose that $a_1 + \cdots + a_A > b_1 + \cdots + b_A$ as integers. Choose α such that $\overline{\alpha}$ has order in $\mathbb{F}^\times$ greater than $(a_1 + \cdots + a_A) - (b_1 + \cdots + b_A)$. Then $\overline{\alpha}^{a_1 + \cdots + a_A} \neq \overline{\alpha}^{b_1 + \cdots + b_A}$. This proves that the eigencharacters of $\overline{\Sigma}$ on M^H are pairwise distinct from the eigencharacters of $\overline{\Sigma}$ on M/M^H. □

Remark 3.9. In the proof of the theorem we used the fact that the coefficients of ξ in the usual basis are all non-negative. This is asserted in [3, proof of Lemma 6.1 (2)]) without a proof. For completeness, here is a proof: Let $\lambda = (a_1, \dots, a_n)$. The irreducible module $F(\lambda)$ is a subquotient of the dual Weyl module $W(\lambda)$. The dual Weyl module is a subquotient of a $\mathbb{Z}$-form of the irreducible $\mathrm{GL}_n(\mathbb{C})$ module Y with highest weight λ modulo an admissible lattice.

Since all these modules are sums of weight spaces, it suffices to show that every weight of Y is obtained from λ by subtracting a linear combination of positive roots with all non-negative coefficients. This follows from [5, Theorem 31.3(b)].

Definition 3.10. Define the filtration

$$M_0 \subset M_1 \subset \cdots \subset M_k = M$$

by setting $M_0 = M^{\mathbf{U}(\mathbb{F})}$, $M_1 =$ the complete inverse image of $(M/M^{\mathbf{U}(\mathbb{F})})^{\mathbf{U}(\mathbb{F})}$ in M, etc. Call this the *H-filtration.*

Because any $\mathbb{F}$-vector space which is a module for a p-group X has a nontrivial fixed point set under X, and because M is finite dimensional over $\mathbb{F}$, the filtration is exhaustive, as intimated by the definition.

The following lemma is clear, because $\mathbf{U}$ is normal in $\mathbf{P}$.

Lemma 3.11. *The H-filtration is stable under $\mathbf{P}(\mathbb{F})$. Its associated graded module is a trivial $\mathbf{U}(\mathbb{F})$-module.*

Lemma 3.12. *Let W be a module for $\mathbf{P}(\mathbb{F})$ which is trivial as a $\mathbf{U}(\mathbb{F})$-module. Let $\Psi_\bullet$ be a resolution of $\mathbb{F}$ by G-modules.*

(1) If $s \in \Sigma$, the map $\psi \otimes_G w \mapsto \psi \otimes_G ws$ provides a well-defined action of Σ on $\Psi \otimes_G W$ which commutes with the differentials and augmentation of Ψ.

(2) This induces an action of Σ on $H_i(G, W)$ for all i.

Proof. The second statement follows from the first. For the first we must check that

$$\psi g \otimes_G wgs = \psi \otimes_G ws$$

for any $g \in G$. By Lemma 3.6, $gs = shg$ for some $h \in \mathbf{U}(\mathcal{O}_\mathfrak{p})$. Then $\mathbf{U}(\mathbb{F})$ acts trivially on W and s normalizes $\mathbf{U}(\mathbb{F})$, so

$$\psi g \otimes_G wgs = \psi g \otimes_G wshg = \psi \otimes_G wsh = \psi \otimes_G wshs^{-1}s = \psi \otimes_G ws.$$

The action clearly commutes with the differentials in Ψ. □

We now form a spectral sequence using the filtration of M. We follow [8, Sections 5.4 and 5.5]. Let $\Psi_\bullet$ be the given resolution of $\mathbb{F}$ by free G-modules. Let A be the complex defined by $A_q = \Psi_q \otimes_G M$. The H-filtration of M induces the filtration $F_\ell A = \Psi \otimes_G M_\ell$ of A. By [8, Theorem 5.5.1], we obtain a spectral sequence

$$E^1_{\ell q} = H_{\ell+q}(\Psi \otimes_G M_\ell/M_{\ell-1}) \Rightarrow H_{\ell+q}(A) = H_{\ell+q}(G, M).$$

In particular, $E^\infty_{0q} = F_0 H_q(A) =$ the image of $H_q(G, M_0)$ in $H_q(G, M)$.

Because $M_\ell/M_{\ell-1}$ is a trivial $\mathbf{U}(\mathcal{O}_\mathfrak{p})$-module, by Lemma 3.12, Σ acts on the E^1 page. The differentials of the spectral sequence are induced by the differentials in Ψ. The Σ-action involves only the second factor of the tensor product, while the differentials involve only the first factor. Therefore the Σ-action commutes with all the differentials of the spectral sequence.

So we have a Σ-action on the spectral sequence and each term is diagonalizable with respect to this action. If we choose a character c of Σ we can project to the c-eigenspace, and get a spectral sequence that converges to the c-eigenspace of the abutment. Let c be the character $c(s_\alpha) = \overline{\alpha}^{a_1+\cdots+a_A}$. By Lemma 3.8, the only terms that have a nonzero projection to this eigenspace are those where $\ell = 0$. Therefore $E^1_{0q} = H_q(\Psi \otimes_G M_0) = H_q(G, M_0)$ survives intact to E^∞_{0q}, which is the image of $H_q(G, M_0)$ in $H_q(G, M)$. In other words, ι_* is injective. QED.

4. Survival to E^∞

We continue the notation from preceding sections, and make the following definition:

Definition 4.1. Let $\Sigma(\mathbb{Z}, N)$ denote the subsemigroup of Σ consisting of s_α for $\alpha \in \mathbb{Z} \cap \mathcal{O}_\mathfrak{p}$ with $\alpha \equiv 1 \pmod N$.

For this section, we will choose our projective resolution F to be the standard resolution of the group generated by G and $\Sigma(\mathbb{Z}, N)$.

Lemma 4.2. *For $s_\alpha \in \Sigma(\mathbb{Z}, N)$ and $f \otimes_H m \in F \otimes_H M^H$, let $(f \otimes_H m) * s_\alpha = fs_\alpha \otimes_H ms_\alpha$. Then this gives a well-defined Q-equivariant action of $\Sigma(\mathbb{Z}, N)$ on $F \otimes_H M^H$.*

Proof. This is an action, if it is well-defined. We must show that if $x \in H$ then

$$fxs_\alpha \otimes_H mxs_\alpha = fs_\alpha \otimes_H ms_\alpha.$$

If $x \in H$ then $fxs_\alpha \otimes_H mxs_\alpha = fs_\alpha s_\alpha^{-1} x s_\alpha \otimes_H ms_\alpha s_\alpha^{-1} x s_\alpha$. But $s_\alpha^{-1} x s_\alpha \in H$, so this equals $fs_\alpha \otimes_H ms_\alpha$.

Now we check that this action is Q-equivariant. The action of $q \in Q$ on $F_* \otimes_H M^H$ is determined by lifting q to $g \in G$ and then sending $f \otimes_H m \mapsto fg \otimes_H mg$. Now

$$(fs_\alpha \otimes_H ms_\alpha)g = fs_\alpha g \otimes_H ms_\alpha g$$

whereas

$$((f \otimes_H m)g)s_\alpha = fgs_\alpha \otimes_H mgs_\alpha = f(s_\alpha g)(s_\alpha g)^{-1}(gs_\alpha) \otimes_H m(s_\alpha g)(s_\alpha g)^{-1}(gs_\alpha).$$

But $g^{-1}s_\alpha^{-1}gs_\alpha \in H$ and hence acts trivially on the tensor product. Indeed, it is easy to see that that $g^{-1}s_\alpha^{-1}gs_\alpha \in \mathbf{U}$. In addition, it has determinant 1 and integer entries so it is in $\mathrm{GL}_n(\mathbb{Z})$. Since Γ is defined by congruence conditions mod N, and s_α is congruent to the identity mod N, we see that $g^{-1}s_\alpha^{-1}gs_\alpha \in \Gamma$. $\square$

Now $\Sigma(\mathbb{Z}, N)$ acts on the double complex

$$C_{*q} = \Phi_* \otimes_Q (F_q \otimes_H M^H)$$

via this action on $F_q \otimes_H M^H$ where $\Sigma(\mathbb{Z}, N)$ acts trivially on Φ_*. It commutes with both differentials, and so gives an action on the spectral sequence

$$E^2_{ij} = H_i(Q, H_j(H, M)) \implies H_{i+j}(G, M)$$

of section 2 arising from the double complex. This action has the following properties.

Lemma 4.3. *Let α be a natural number prime to p, so that $s_\alpha \in \Sigma(\mathbb{Z}, N)$.*

(1) The $*$-action of s_α on $H_q(H, M^H) = \wedge^q H \otimes M^H$ is the tensor product of the action on $\wedge^q H$ induced by $h \mapsto h^\alpha$ and the usual action of s_α on M^H.

(2) This action commutes with the Q-action on $H_q(H, M^H)$.

(3) The $*$-action of s_α on $H_q(H, M^H)$, and therefore on $\oplus_r H_r(Q, H_q(H, M^H))$, is multiplication by the scalar $\overline{\alpha}^{q+a_1+\cdots+a_A}$.

Proof. (1) By Corollary 3.8(1), as a $\Sigma(\mathbb{Z}, N)$-module, $M^H \cong \mathbb{F}_c^m$ for some m, where $c(s_\alpha) = (\overline{\alpha})^{a_1+\cdots+a_A}$. Without loss of generality we may take $m = 1$. The Pontryagin product is natural, so we may take $q = 1$. If we compute the homology of H using the resolution F, the $*$-action of s_α on the chains induces the natural action of s_α on $H_1(H, \mathbb{F}_c)$. Here, s_α acts by right conjugation on H and on the coefficients via c.

The action of s_α on H is given by the formula

$$\begin{bmatrix} \alpha^{-1} I_A & 0 \\ 0 & I_B \end{bmatrix} \begin{bmatrix} I_A & 0 \\ u & I_B \end{bmatrix} \begin{bmatrix} \alpha I_A & 0 \\ 0 & I_B \end{bmatrix} = \begin{bmatrix} I_A & 0 \\ \alpha u & I_B \end{bmatrix} = \begin{bmatrix} I_A & 0 \\ u & I_B \end{bmatrix}^\alpha .$$

Translated into homology, which we will write additively, the action of s_α on $H_1(H, \mathbb{F})$ is multiplication by $\overline{\alpha}$. Hence the action of s_α on $H_q(H, \mathbb{F}_c)$ is as stated in the lemma.

(2) The isomorphism $H_1(H, \mathbb{F}_c) \to H \otimes \mathbb{F}_c$ is functorial and therefore it is an isomorphism of Q-modules. Since the action of s_α is just multiplication by a scalar, it commutes with the Q-action.

(3) Since s_α is a semi-scalar matrix, its conjugation action on Q is trivial. So the action of s_α on $\oplus_r H_r(Q, H_q(H, M^H))$ is only through its action on $H_q(H, M^H)$. Hence (3) follows from (1). □

Theorem 4.4. *Suppose $p > d+1$. Let $z \in E^2_{jd} = H_j(Q, H_d(H, M^H))$ be a nonzero class. Then z persists to a nonzero class in E^∞_{jd}.*

Proof. Recall that the spectral sequence arises from the double complex

$$C_{*q} = \Phi_* \otimes_Q (F_q \otimes_H M^H),$$

where we have chosen F to be the standard resolution of the group generated by G and $\Sigma(\mathbb{Z}, N)$. For $\alpha \in \mathcal{O}_\mathfrak{p}^\times$, s_α acts on M via its reduction modulo $\mathfrak{p}$.

Now suppose $z \in E^2_{jd}$ does not survive to E^∞. By Lemma 4.3, for $s_\alpha \in \Sigma(\mathbb{Z}, N)$,

$$z * s_\alpha = \overline{\alpha}^{d+a_1+\cdots+a_A} z.$$

Recall that z is in the kernel of all the higher differentials. So if $z \in E^2_{jd}$ does not survive to E^∞, then for some $\ell \geq 2$, there exists w such that $z_\ell = d_\ell(w)$. Here, z_ℓ is the image of z in the ℓ page of the spectral sequence, and w is the image of some $W \in (E)^2_{j+\ell, d-\ell+1} = H_{j+\ell}(Q, H_{d-\ell+1}(H, M))$ in the kernel of $d_2, \ldots, d_{\ell-1}$.

By Lemma 4.3,

$$W * s_\alpha = \overline{\alpha}^{d-\ell+1+a_1+\cdots+a_A} W.$$

Because the differentials commute with the action of $\Sigma(\mathbb{Z}, N)$,

$$w * s_\alpha = \overline{\alpha}^{d-\ell+1+a_1+\cdots+a_A} w.$$

On the other hand,

$$w * s_\alpha = \overline{\alpha}^{d+a_1+\cdots+a_A} w$$

since z is the image of w under one of the differentials.

Choose α so that $\overline{\alpha}$ generates $(\mathbb{Z}/p)^\times$. Since $w \neq 0$, we must have

$$-\ell + 1 \equiv 0 \pmod{p-1}$$

i.e. $(p-1)|(\ell-1)$. But $2 \leq \ell \leq d+1$. Therefore $p-1 \leq \ell - 1 \leq d$, i.e. $p \leq d+1$. This contradicts the hypothesis. □

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