HOMEWORK 6

(1) Let R be a Euclidean domain. Prove that $u \in R$ is a unit if and only if $\delta(u) \leq \delta(a)$ for all $a \in R - \{0\}$.

(2) Prove that any field is a Euclidean domain with $\delta(a) = 0$.

(3) Let $f = x^5 + 2x^4 + 3x^3 + 3x^2 + 2x + 1$ and $g = x^5 + 3x^4 + 4x^3 + 4x^2 + 2x + 1$ be elements of $\mathbb{Q}[x]$. Determine a generator for the ideal $\langle f, g \rangle$ in $\mathbb{Q}[x]$.

(4) Prove that the ring $\mathbb{Z}[i] = \{a + bi : a, b \in \mathbb{Z}\}$ is a Euclidean domain with $\delta(a + bi) = a^2 + b^2$. (Hint: $\mathbb{Q}[i]$ is a field, so you can divide there. To prove the division algorithm, divide in $\mathbb{Q}[i]$, and round off the quotient to the nearest integer–then prove that the remainder is small enough.)