Example: Let $f(x) = x^3 + bx^2 + cx + d$ have roots α_1, α_2, and α_3. Find the polynomial $g(x)$ that has roots $\alpha_1 + \alpha_2$, $\alpha_1 + \alpha_3$, and $\alpha_2 + \alpha_3$.

Solution: Writing

$$f(x) = x^3 + bx^2 + cx + d = (x - \alpha_1)(x - \alpha_2)(x - \alpha_3)$$

we find that

$$f(x) = x^3 - \sigma_1 x^2 + \sigma_2 x^2 - \sigma_3$$

where we write σ_i as shorthand for $\sigma_i(\alpha_1, \alpha_2, \alpha_3)$. Hence $b = -\sigma_1$, $c = \sigma_2$ and $d = \sigma_3$.

Now $g(x) = (x - (\alpha_1 + \alpha_2))(x - (\alpha_1 + \alpha_3))(x - (\alpha_2 + \alpha_3))$.

Expanding this out, we find that

$$\begin{aligned} g(x) = x^3 &- 2(\alpha_1 + \alpha_2 + \alpha_3)x^2 \\ &+ (\alpha_1^2 + \alpha_2^2 + \alpha_3^2 + 3\alpha_1\alpha_2 + 3\alpha_1\alpha_3 + 3\alpha_2\alpha_3)x \\ &- (\alpha_1^2\alpha_2 + \alpha_1^2\alpha_3 + \alpha_2^2\alpha_3 + \alpha_2^2\alpha_1 + \alpha_3^2\alpha_1 + \alpha_3^2\alpha_2 + 2\alpha_1\alpha_2\alpha_3) \end{aligned}$$

We note that the coefficient of the x^2 term is just $-2\sigma_1$, or $2b$.

The coefficient of the x term is $\sigma_1^2 - 2\sigma_2 + 3\sigma_2 = \sigma_1^2 + \sigma_2$. (As we showed in class, $\alpha_1^2 + \alpha_2^2 + \alpha_3^2 = \sigma_1^2 - 2\sigma_2$.) Hence, this coefficient is $\sigma_1^2 + \sigma_2 = (-b)^2 + c = b^2 + c$.

Finally, to simplify the constant coefficient, we use the algorithm from class today. Namely, find the leading term of

$$h = \alpha_1^2\alpha_2 + \alpha_1^2\alpha_3 + \alpha_2^2\alpha_3 + \alpha_2^2\alpha_1 + \alpha_3^2\alpha_1 + \alpha_3^2\alpha_2 + 2\alpha_1\alpha_2\alpha_3$$

(I have removed the minus sign for convenience) to be $\alpha_1^2\alpha_2 = \alpha_1^2\alpha_2^1\alpha_3^0$. As described in class, the leading term of $\sigma_1^{(2-1)}\sigma_2^{(1-0)}\sigma_3^0$ is also $\alpha_1^2\alpha_2^1\alpha_3^0$. Then $h - \sigma_1\sigma_2$ will have a smaller leading term. Doing the arithmetic (on a computer, of course), we find that

$$h - \sigma_1\sigma_2 = -\alpha_1\alpha_2\alpha_3 = -\sigma_3.$$

Hence, $h = \sigma_1\sigma_2 - \sigma_3 = (-b)c - (-d) = -bc + d$.

Finally,

$$\begin{aligned} g(x) &= x^3 + 2bx^2 + (b^2 + c)x - h \\ &= x^2 + 2bx^2 + (b^2 + c)x + (bc - d). \end{aligned}$$

To check this, we let $f(x) = (x-1)(x-2)(x-3) = x^3 - 6x^2 + 11x - 6$ have roots $1, 2, 3$. The coefficients are $b = -6$, $c = 11$, and $d = -6$. Then

$$\begin{aligned} g(x) &= x^2 + 2bx^2 + (b^2 + c)x + (bc - d) \\ &= x^3 + 2(-6)x^2 + ((-6)^2 + 11)x + ((-6)(11) - (-6)) \\ &= x^3 - 12x^2 + 47x - 60 \\ &= (x-3)(x-4)(x-5) \end{aligned}$$

has roots $3 = 1 + 2$, $4 = 1 + 3$ and $5 = 2 + 3$, as desired.