**MATH 473**
**WINTER 2019**
**HOMEWORK 1**

1. Suppose that $G$ is a group and $H$ is a normal subgroup of $G$ of prime index. Let $K$ be a subgroup of $G$. Prove that
$$\left|\frac{K}{K \cap H}\right| = \begin{cases} 1 & \text{if } K \subseteq H, \\ [G : H] & \text{if } K \nsubseteq H. \end{cases}$$

2. Let $G$ be a finite cyclic group, and let $x, y \in G$. Prove that if the order of $y$ divides the order of $x$, then $y$ is a power of $x$.

3. Let
$$G = D_8 = \langle a, b : a^4 = b^2 = 1, b^{-1}ab = a^{-1} \rangle$$
and
$$H = Q_8 = \langle c, d : c^4 = 1, c^2 = d^2, d^{-1}cd = c^{-1} \rangle.$$
   (a) Let $x = (1,2)$ and $y = (3,4)$ be permutations in $S_4$, and let $K$ be the subgroup of $S_4$ generated by $x$ and $y$. Define
   $$\phi : G \to K \quad \text{by} \quad (a^r b^s)\phi = x^r y^s$$
   and
   $$\psi : H \to K \quad \text{by} \quad (c^r d^s)\psi = x^r y^s$$
   for all $r, s \in \mathbb{Z}$. Prove that $\phi$ and $\psi$ are well-defined homomorphisms, and find $\ker \phi$ and $\ker \psi$.
   (b) Let
   $$X = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \quad \text{and} \quad Y = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$
   and let $L = \langle X, Y \rangle \subset \mathrm{GL}(2, \mathbb{C})$. Define
   $$\lambda : G \to L \quad \text{by} \quad (a^r b^s)\lambda = X^r Y^s$$
   and
   $$\mu : H \to L \quad \text{by} \quad (c^r d^s)\mu = X^r Y^s,$$
   for $r, s \in \mathbb{Z}$.
   Prove that only one of $\lambda$ and $\mu$ is a well-defined function, and that it is not only well-defined, but is an isomorphism.

4. Find elements $A$ and $B$ of $\mathrm{GL}(2, \mathbb{C})$ such that $A$ has order 8, $B$ has order 2, and
$$B^{-1}AB = A^{-1}.$$
Show that the group $\langle A, B \rangle$ has order 16.