

MATH 473
WINTER 2013
HOMEWORK 2

1. Let V be a vector space, and let U and W be subspaces of V . Prove that $V = U \oplus W$ if and only if $V = U + W$ and $U \cap W = 0$.
2. Suppose that U_1, \dots, U_r are subspaces of V with $V = U_1 \oplus \dots \oplus U_r$, and that each U_i has a basis B_i . Prove that the set

$$B = \bigcup_{i=1}^r B_i$$

is a basis for V .

3. Give an example of a vector space V and an endomorphism ϑ of V such that $V = \text{Im } \vartheta \oplus \text{Ker } \vartheta$, but ϑ is not a projection.
4. Suppose that ϑ is an endomorphism of a vector space V over \mathbb{C} such that $\vartheta^3 = 1_V$. Show that $V = U_0 \oplus U_1 \oplus U_2$, where

$$\begin{aligned} U_0 &= \{v \in V : v\vartheta = v\}, \\ U_1 &= \{v \in V : v\vartheta = \zeta v\}, \\ U_2 &= \{v \in V : v\vartheta = \zeta^2 v\}, \end{aligned}$$

where $\zeta = e^{2\pi i/3}$ is a cube root of 1.

Hints: Note that $\zeta^3 = 1$ and $1 + \zeta + \zeta^2 = 0$. If $v \in V$ and $w_1 = \frac{1}{3}(v + \zeta^2 v\vartheta + \zeta v\vartheta^2)$, what is $w_1\vartheta$? Does w_1 live in one of U_0 , U_1 , or U_2 ? Can you find a w_0 and a w_2 so that $v = w_0 + w_1 + w_2$?