MATH 473
WINTER 2019
HOMEWORK 8

1. Let $G = \langle (1\ 2), (3\ 4) \rangle \in S_4$ and let V be the permutation FG-module with natural basis $\{v_1, v_2, v_3, v_4\}$.

   Classify each of the following subspaces of V as an "irreducible submodule," a "reducible submodule," or "not a submodule."

   $U_1 = \text{span}(v_1 + v_2)$, $U_2 = \text{span}(v_1 + v_2 + 2v_3 + 2v_4)$,
   $U_3 = \text{span}(v_1, v_2)$, $U_4 = \text{span}(v_3, v_4)$,
   $U_5 = \text{span}(v_1 - v_2, v_3 - v_4)$, $U_6 = \text{span}(v_1 - v_2)$,
   $U_7 = \text{span}(v_1 + 2v_2 + 2v_3 + v_4)$, $U_8 = \text{span}(v_3 - v_4)$,
   $U_9 = \text{span}(7v_1 + 7v_2 + 5v_3 + 5v_4)$, $U_{10} = \text{span}(v_1 + v_2 + v_3 + v_4)$.

2. Find a subset S of the irreducible submodules in problem 1 such that V is the direct sum of the submodules in S.

3. Let $G = \langle (1\ 2), (3\ 4) \rangle \in S_4$ and let V be the permutation FG-module with natural basis $\{v_1, v_2, v_3, v_4\}$. Let $U = \text{span}(v_1 + v_2, v_3 + v_4)$. Then U is an FG-submodule of V. Using the procedure described in Maschke's theorem, find an FG-submodule W such that $V = U \oplus W$.

   Hint: A reasonable choice for W_0 would be $\text{span}(v_2, v_4)$.

4. Let G be a finite group, and let $\rho : G \to \text{GL}(2, \mathbb{C})$ be a representation of G. Assume that for some $g, h \in G$, the matrices $g\rho$ and $h\rho$ do not commute with each other. Prove that ρ is an irreducible representation.

   (*Hint:* The fact that the representation is two-dimensional is necessary.)