MATH 473
FALL 2019
HOMEWORK 1

1. Suppose that G is a group and H is a normal subgroup of G of prime index. Let K be a subgroup of G. Prove that
$$\left|\frac{K}{K\cap H}\right| = \begin{cases} 1 & \text{if } K \subseteq H, \\ [G:H] & \text{if } K \not\subseteq H. \end{cases}$$

2. Let G be a finite cyclic group, and let $x, y \in G$. Prove that if the order of y divides the order of x, then y is a power of x.

3. Let
$$G = D_8 = \langle a, b : a^4 = b^2 = 1, b^{-1}ab = a^{-1}\rangle$$
and
$$H = Q_8 = \langle c, d : c^4 = 1, c^2 = d^2, d^{-1}cd = c^{-1}\rangle.$$
   (a) Let $x = (1,2)$ and $y = (3,4)$ be permutations in S_4, and let K be the subgroup of S_4 generated by x and y. Define
   $$\phi : G \to K \quad \text{by} \quad (a^r b^s)\phi = x^r y^s$$
   and
   $$\psi : H \to K \quad \text{by} \quad (c^r d^s)\psi = x^r y^s$$
   for all $r, s \in \mathbb{Z}$. Prove that ϕ and ψ are well-defined homomorphisms, and find $\ker\phi$ and $\ker\psi$.
   (b) Let
   $$X = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \quad \text{and} \quad Y = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$
   and let $L = \langle X, Y\rangle \subset \mathrm{GL}(2, \mathbb{C})$. Define
   $$\lambda : G \to L \quad \text{by} \quad (a^r b^s)\lambda = X^r Y^s$$
   and
   $$\mu : H \to L \quad \text{by} \quad (c^r d^s)\mu = X^r Y^s,$$
   for $r, s \in \mathbb{Z}$.
   Prove that only one of λ and μ is a well-defined function, and that it is not only well-defined, but is an isomorphism.

4. Find elements A and B of $\mathrm{GL}(2, \mathbb{C})$ such that A has order 8, B has order 2, and
$$B^{-1}AB = A^{-1}.$$
Show that the group $\langle A, B\rangle$ has order 16.