MATH 473
FALL 2019
HOMEWORK 11

1. Let G be a finite group, and let $\rho : G \to \mathrm{GL}(n, \mathbb{C})$ be an irreducible representation. Prove that if $z \in Z(G)$, then $z\rho$ is a scalar matrix (a scalar multiple of the identity matrix).

   Hint: There are two different ways to do this, one uses a corollary from chapter 9, the other uses a proposition from chapter 9.

2. Suppose that V is a non-zero $\mathbb{C}G$-module such that $V = U_1 \oplus U_2$, U_1 and U_2 are $\mathbb{C}G$-modules and there is a $\mathbb{C}G$-isomorphism $\vartheta : U_1 \to U_2$. Prove that there is a $\mathbb{C}G$-submodule U of V that is not equal to either U_1 or U_2, but is isomorphic to both of them.

3. Let $G = D_8 = \langle a, b : a^4 = b^2 = e, b^{-1}ab = a^{-1} \rangle$.
   (a) Prove that the span of the vectors $1 - ia - a^2 + ia^3$ and $b - iab - a^2b + ia^3b$ is an irreducible $\mathbb{C}G$-submodule of $\mathbb{C}G$.
   (b) Find a second two-dimensional irreducible $\mathbb{C}G$-submodule of $\mathbb{C}G$.

4. Find four nonisomorphic one-dimensional irreducible $\mathbb{C}G$-submodules of $\mathbb{C}G$ when $G = D_8$. Be sure to prove that each one is actually a submodule and that they are not isomorphic.

   Hint: One of them is spanned by

$$(1 + b)(1 - a + a^2 - a^3) = (1 - a + a^2 - a^3)(1 + b).$$