MATH 473
FALL 2019
HOMEWORK 2

1. Let V be a vector space, and let U and W be subspaces of V. Prove that $V = U \oplus W$ if and only if $V = U + W$ and $U \cap W = 0$.

2. Suppose that $U_1, \dots, U_r$ are subspaces of V with $V = U_1 \oplus \cdots \oplus U_r$, and that each U_i has a basis B_i. Prove that the set
$$B = \bigcup_{i=1}^{r} B_i$$
is a basis for V.

3. Give an example of a vector space V and an endomorphism ϑ of V such that $V = \operatorname{Im}\vartheta \oplus \operatorname{Ker}\vartheta$, but ϑ is not a projection.

4. Suppose that ϑ is an endomorphism of a vector space V over $\mathbb{C}$ such that $\vartheta^3 = 1_V$. Show that $V = U_0 \oplus U_1 \oplus U_2$, where
$$U_0 = \{v \in V : v\vartheta = v\},$$
$$U_1 = \{v \in V : v\vartheta = \zeta v\},$$
$$U_2 = \{v \in V : v\vartheta = \zeta^2 v\},$$
where $\zeta = e^{2\pi i/3}$ is a cube root of 1.

   Hints: Note that $\zeta^3 = 1$ and $1 + \zeta + \zeta^2 = 0$. If $v \in V$ and $w_1 = \frac{1}{3}(v + \zeta^2 v\vartheta + \zeta v\vartheta^2)$, what is $w_1\vartheta$? Does w_1 live in one of U_0, U_1, or U_2? Can you find a w_0 and a w_2 so that $v = w_0 + w_1 + w_2$?