MATH 473
FALL 2019
HOMEWORK 30

(1) Let H and K be subgroups of G such that $HK = G$. Define the map $f : H \times K \to G$ by $f(h,k) = hk$. Prove that for every $g \in G$, there are exactly $|H \cap K|$ elements (h,k) with $f(h,k) = g$. Conclude that

$$|G| \cdot |H \cap K| = |H| \cdot |K|.$$

(2) Let H and K be subgroups of G such that $HK = G$. Let ψ be a character of H. Prove that

$$(\psi \uparrow G){\downarrow}K = (\psi{\downarrow}(H \cap K)) \uparrow K$$

Hint: By problem 1, we may replace y by $(hk)^{-1}$ and replace

$$\sum_{y \in G}$$

by

$$\frac{1}{|H \cap K|} \sum_h \sum_k .$$

(3) Prove that a complex number λ that is the root of a monic polynomial with integer coefficients is an algebraic integer. (*Hint: Look up the "companion matrix"*)

(4) Let G be a nonabelian group of order p^3. Determine the degrees of the irreducible characters of G, and how many irreducible characters of each degree there are.

Hint: The following group theory facts may be helpful (although the proof can be done without them).
a) If G is a group of order p^n, then the center $Z(G)$ is nontrivial.
b) If $G/Z(G)$ is cyclic, then G is abelian.