MATH 473
FALL 2019
HOMEWORK 31

1. Let p and q be primes with $q > p$, and let G be a nonabelian group of order pq.
   (a) Determine the degrees of the irreducible characters of G.
   (b) How many one dimensional characters does G have? (Prove your answer.)
   (c) Prove that $p|(q-1)$.
   (d) Determine how many conjugacy classes G has.

2. Prove that every group of order 35 is abelian.

3. Let G be a group of odd order, and let 1_G be the trivial character of G.
   (a) Prove that the only element $g \in G$ with $g = g^{-1}$ is the identity.
   (b) Suppose that χ is an irreducible character of G with $\chi = \bar{\chi}$. Prove that
$$\langle \chi, 1_G \rangle = \frac{1}{|G|}(\chi(1) + 2a)$$
   for some algebraic integer a.
   (c) Prove that every irreducible character χ of G with $\chi = \bar{\chi}$ must satisfy $\chi = 1_G$.
   (d) Prove that the number of conjugacy classes of G is odd.

4. Let G be a group, and let φ be a character of G such that $\varphi(g) = \varphi(h)$ for all nonidentity elements g and h of G. Let 1_G be the trivial character of G, and χ_{reg} the regular character of G.
   (a) Show that $\varphi = a1_G + b\chi_{\text{reg}}$ for some $a, b \in \mathbb{C}$.
   (b) Show that $a + b$ and $a + b|G|$ are integers.
   (c) Show that if χ is a nontrivial irreducible character of G, then $b\chi(e)$ is an integer.
   (d) Show that a and b are integers.