MATH 473
FALL 2019
HOMEWORK 34

1. Let V be an $\mathbb{R}G$-module, and let β be a G-invariant symmetric bilinear form on V. Assume that there exist $u, v \in V$ such that $\beta(u,u) > 0$ and $\beta(v,v) < 0$. Let β_1 be a G-invariant symmetric bilinear form on V with $\beta_1(x,x) > 0$ for all nonzero $x \in V$. We then know that we can choose a basis $\{f_1, \ldots, f_n\}$ of V with
$$\beta_1(f_i, f_j) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$
   (a) Let $b_{ij} = \beta(f_i, f_j)$. Prove that $B = (b_{ij})$ is symmetric. Hence, by a well known theorem, B is orthogonally diagonalizable (there is a real matrix $Q = (q_{ij})$ with $QQ^T = I$ such that QBQ^T is diagonal).
   (b) For $1 \leq i \leq n$, set
$$e_i = \sum_j q_{ij} f_j.$$
   Prove that $\beta(e_r, e_s) = 0$ if $r \neq s$, and that
$$\beta_1(e_r, e_s) = \begin{cases} 1 & \text{if } r = s, \\ 0 & \text{if } r \neq s. \end{cases}$$
   (c) Prove that for at least one e_i, we have $\beta(e_i, e_i) > 0$.
   (d) Prove that for at least one e_j, we have $\beta(e_j, e_j) < 0$.

2. Let U be an $\mathbb{R}G$-module with basis $v_1, \ldots, v_n$, and corresponding representation $\rho : G \to \mathrm{GL}(n, \mathbb{R})$. Let V be the $\mathbb{C}G$-module with the same basis, and G-action given by the same representation. Let γ be a nonzero symmetric G-invariant bilinear form on U.

   Define, for $\lambda_i, \mu_j \in \mathbb{C}$,
$$\hat{\gamma}\left(\sum_i \lambda_i v_i, \sum_j \mu_j v_j\right) = \sum_i \sum_j \lambda_i \mu_j \gamma(v_i, v_j).$$
   Prove that $\hat{\gamma}$ is a nonzero symmetric G-invariant bilinear form on V.

3. Let χ be a character of a finite group G. Assume that χ can be realized over $\mathbb{R}$. Prove that there is a $\mathbb{C}G$-module V with character χ such that there is a nonzero symmetric G-invariant bilinear form on V.

4. Let χ be an irreducible character of G. Determine the possible values of
$$\langle \chi^2, 1_G \rangle_G,$$
   where 1_G represents the trivial character of G.