MATH 473
FALL 2019
HOMEWORK 5

1. Let $G = S_n$ and let V be an m-dimensional vector space over F (where $F = \mathbb{R}$ or $\mathbb{C}$. Then (see problem 4.2) V becomes an FG-module if we define
$$vg = \begin{cases} v & \text{if } g \text{ is an even permutation,} \\ -v & \text{if } g \text{ is an odd permutation.} \end{cases}$$
Let B be a basis for V. Describe the representation ρ corresponding to the FG-module V and the basis B.

2. Consider $G = C_4$ as the subgroup of S_4 generated by the permutation $(1\ 2\ 3\ 4)$. Describe the representation corresponding to the permutation module of G over $F = \mathbb{C}$ with respect to the natural basis.

3. With the same notation as problem 2, let $\{v_1, v_2, v_3, v_4\}$ be the natural basis. Describe the representation of G corresponding to the permutation module of G over $F = \mathbb{C}$ with respect to the basis
$\{v_1 + v_2 + v_3 + v_4, v_1 + iv_2 - v_3 - iv_4, v_1 - v_2 + v_3 - v_4, v_1 - iv_2 - v_3 + iv_4\}$.

4. Let n be a natural number, and let G be a subgroup of S_n. Let V be the permutation module for G over $\mathbb{C}$, with natural basis $\{v_1, \dots, v_n\}$.
   (a) Prove that $U = \langle v_1 + v_2 + \cdots + v_n \rangle$ is a submodule of V.
   (b) Prove that
   $$W = \left\{ \sum_{i=1}^{n} \lambda_i v_i : \lambda_i \in \mathbb{C}, \sum_{i=1}^{n} \lambda_i = 0 \right\}$$
   is a submodule of V.
   (c) Prove that $V = U \oplus W$.