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Local corrections of discriminant bounds and small degree extensions of quadratic base fields

Sharon Brueggeman*
Mathematics Department
University of Tennessee at Chattanooga
Chattanooga, TN 37403
Sharon-Brueggeman@utc.edu

Darrin Doud†
Department of Mathematics
Brigham Young University
Provo, UT 84602
doud@math.byu.edu

Using analytic techniques of Odlyzko and Poitou, we create tables of lower bounds for discriminants of number fields, including local corrections for ideals of known norm. Comparing the lower bounds found in these tables with upper bounds on discriminants of number fields obtained from calculations involving differents, we prove the nonexistence of a number of small degree extensions of quadratic fields having limited ramification. We note that several of our results require the locally corrected bounds.

Keywords: Discriminant bounds; local corrections.

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1. Introduction

Over the past decade, there have been many articles describing computer searches to find examples of number fields which have specified prime ramification. In fact, all number fields of degrees 5 and 6 [4] and degree 7 [1] which are ramified at a single small prime p and where $p \leq 7$ have been found. Not one has a nonsolvable Galois group. Lesseni Sylla [11,12] has shown there are no nonsolvable examples arising from number fields of degree 8 or 9.

In this paper, we investigate degree m extensions ($5 \leq m \leq 9$) of quadratic base fields using discriminant bounds. We make special use of local corrections and

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ramification structures. In Section 2, we describe methods of Odlyzko [7], Poitou [8], and Selmane [9] to produce large lower bounds. In Section 3, we describe a method of producing upper bounds using the different. Finally, in Section 4, we present our nonexistence results. The choices of quadratic fields and degrees for these nonexistence results were guided by the desire to study nonsolvable extensions of $\mathbb{Q}$ ramified at only one prime.

Throughout this paper, we will use the following notation. K will be a number field of degree n over $\mathbb{Q}$, with r_1 real places and $2r_2$ complex places. The norm of a prime ideal $\mathfrak{P}$ of K will be denoted by $N\mathfrak{P}$. For any field F, we denote the absolute discriminant of the field by d_F.

2. Analytic lower bounds on discriminants

2.1. *Weil's explicit formula*

A valuable tool for obtaining useful lower bounds on discriminants of number fields is Weil's explicit formula for the zeta function of a number field. We use this formula in the following form:

Proposition 2.1. *[8, pg 6-06] Let $K/\mathbb{Q}$ be a number field with discriminant d_K. Let $F(x)$ be a continuous even real-valued function on the real line satisfying*

(1) there exists $\epsilon > 0$ such that $F(x)\exp((1/2+\epsilon)x)$ is integrable,
(2) there exists $\epsilon > 0$ such that $F(x)\exp((1/2+\epsilon)x)$ is of bounded variation,
(3) the function $(F(0)-F(x))/x$ is of bounded variation,

and let

$$\Phi(s) = \int_{-\infty}^{\infty} F(x)\exp((s-1/2)x)\,dx.$$

We have the following equality:

$$\begin{aligned} F(0)\left(\log|d_k| - n(\gamma+\log 8\pi) - r_1\frac{\pi}{2}\right) = {} & \sum_{\rho}\Phi(\rho) - \Phi(0) - \Phi(1) \\ & + 2\sum_{j=1}^{\infty}\sum_{\mathfrak{P}} \frac{\log(N\mathfrak{P})}{(N\mathfrak{P})^{j/2}} F(j\log(N\mathfrak{P})) \\ & - r_1\int_0^{\infty} \frac{F(0)-F(x)}{2\cosh(x/2)}\,dx \\ & - n\int_0^{\infty} \frac{F(0)-F(x)}{2\sinh(x/2)}\,dx, \end{aligned}$$

where ρ runs over the zeros of the Dedekind zeta function of K and $\mathfrak{P}$ runs over the prime ideals of K.

We now take $F(x)$ to be a function with $F(0) = 1$, use the fact that $\Phi(0)+\Phi(1) = 2\int_{-\infty}^{\infty} F(x)\cosh(x/2)\,dx = 4\int_0^{\infty} F(x)\cosh(x/2)\,dx$, and solve for $\log|d_K|$ to obtain the following proposition:

Proposition 2.2. *Let $F(x)$ be a continuous even real-valued function on the real line satisfying conditions (1), (2), (3), of Prop. 2.1, and let $\Phi(s) = \int_{-\infty}^{\infty} F(x)\exp((s-1/2)x)\,dx$. If $F(0)=1$, then*

$$\begin{aligned}\log|d_K| = {} & r_1\frac{\pi}{2} + n(\gamma+\log 8\pi)\\ & - r_1\int_0^\infty \frac{1-F(x)}{2\cosh(x/2)}\,dx - n\int_0^\infty \frac{1-F(x)}{2\sinh(x/2)}\,dx - 4\int_0^\infty F(x)\cosh(x/2)\,dx\\ & + 2\sum_{j=1}^{\infty}\sum_{\mathfrak{P}} \frac{\log(N\mathfrak{P})}{(N\mathfrak{P})^{j/2}} F(j\log(N\mathfrak{P})) + \sum_{\rho}\Phi(\rho).\end{aligned}$$

2.2. *Choosing the function $F(x)$*

To obtain a lower bound on $\log|d_K|$ we wish to guarantee that the two sums (over prime ideals and over roots of ζ_K) are nonnegative. For this to happen, we will require that $F(x)$ be even and nonnegative for all real x, and that $\Phi(s)$ have nonnegative real part everywhere in the critical strip. This is equivalent [8,7] to choosing $F(x)$ of the form

$$F(x) = \frac{f(x\sqrt{y})}{\cosh(x/2)},$$

where $f(x)$ is even and nonnegative with nonnegative Fourier transform, and y is a parameter.

Assuming that the function $F(x)$ is of this form, relaxing the conditions on $F(x)$ slightly (as described by Poitou [8, 6-08]), and performing some simple algebraic simplifications, we obtain the following proposition.

Proposition 2.3. *[8, Prop. 5] Let K be a number field of degree n with r_1 real embeddings. Let $f(x)$ be a continuous even nonnegative function with $f(0)=1$, satisfying*

(1) the integral $\int_0^\infty f(x)\,dx$ converges,
(2) the functions $f(x)/cosh(x/2)$ and $(1-f(x))/x$ are of bounded variation,
(3) the function $f(x)$ has nonnegative Fourier transform.

Then

$$\begin{aligned}\log|d_K| > {} & r_1 + n(\gamma+\log 4\pi)\\ & - r_1\int_0^\infty \frac{1-f(x\sqrt{y})}{2\cosh^2(x/2)}\,dx - n\int_0^\infty \frac{1-f(x\sqrt{y})}{\sinh x}\,dx - 4\int_0^\infty f(x\sqrt{y})\,dx\\ & + 4\sum_{j=1}^{\infty}\sum_{\mathfrak{P}} \frac{\log(N\mathfrak{P})}{1+(N\mathfrak{P})^j} f(j\log(N\mathfrak{P})\sqrt{y}),\end{aligned}$$

where $\mathfrak{P}$ runs through the prime ideals of K.

The best known choice for $f(x)$ satisfying the conditions of the proposition was constructed by Luc Tartar [8, pg. 6-13], [7], and is given by

$$f(x) = \left(\frac{3}{x^3} (\sin x - x \cos x) \right)^2 .$$

One checks easily that for this choice of $f(x)$,

$$\int_0^\infty f(x\sqrt{y})\, dx = \frac{3\pi}{5\sqrt{y}}.$$

We will write

$$I(y) = r_1 \int_0^\infty \frac{1 - f(x\sqrt{y})}{2\cosh^2(x/2)}\, dx + n \int_0^\infty \frac{1 - f(x\sqrt{y})}{\sinh x}\, dx \tag{2.1}$$

and

$$C(\mathfrak{P}, y) = 4 \sum_{j=1}^\infty \frac{\log(N\mathfrak{P})}{1 + (N\mathfrak{P})^j} f(j \log(N\mathfrak{P})\sqrt{y}). \tag{2.2}$$

Proposition 2.3 then states:

$$\log |d_K| \geq r_1 + n(\gamma + \log 4\pi) - \frac{12\pi}{5\sqrt{y}} - I(y) + \sum_{\mathfrak{P}} C(\mathfrak{P}, y).$$

Using the fact that the sum over all $\mathfrak{P}$ has positive summands we obtain the inequality

$$\log |d_K| \geq r_1 + n(\gamma + \log 4\pi) - \frac{12\pi}{5\sqrt{y}} - I(y)$$

valid for all number fields K with degree n and r_1 real embeddings and all positive y.

Under the assumption of the Generalized Riemann Hypothesis (namely that all the roots of ζ_K have real part $\frac{1}{2}$), we wish to choose a positive function $F(x)$ such that the real part of $\Phi(\rho)$ is positive for each complex number with real part 1/2 (hence, for each root ρ of ζ_K). For this purpose, Poitou [8, pg 6-09] suggests the use of functions of the form $G(x\sqrt{y})$ with

$$G(x) = \begin{cases} (1 - |x|) \cos |\pi x| + \frac{1}{\pi} \sin |\pi x|, & \text{if } x \in [-1, 1] \\ 0, & \text{otherwise,} \end{cases}$$

and y a positive parameter.

For this choice of G we will write

$$\begin{aligned} J(y) =& r_1 \int_0^\infty \frac{1 - G(x\sqrt{y})}{2\cosh(x/2)}\, dx + n \int_0^\infty \frac{1 - G(x\sqrt{y})}{2\sinh(x/2)}\, dx \\ &+ 4 \int_0^\infty G(x\sqrt{y}) \cosh(x/2)\, dx, \end{aligned} \tag{2.3}$$

and

$$B(\mathfrak{P}, y) = 2\sum_{j=1}^{\infty} \frac{\log(N\mathfrak{P})}{(N\mathfrak{P})^{j/2}} G(j\log(N\mathfrak{P})\sqrt{y}). \tag{2.4}$$

We may then write the explicit formula as

$$\log|d_k| = r_1\frac{\pi}{2} + n(\gamma + \log 8\pi) - J(y) + \sum_{\mathfrak{P}} B(\mathfrak{P}, y) + \sum_{\rho} \Phi(\rho),$$

where $\Phi(s) = \int_{-\infty}^{\infty} G(x)\exp((s-1/2)x)\,dx$. Since both of the two sums in the formula above are positive, we obtain the following inequality valid for positive y under the assumption of the GRH,

$$\log|d_k| \geq r_1\frac{\pi}{2} + n(\gamma + \log 8\pi) - J(y).$$

2.3. *Local Corrections*

If we know that the number field K contains a prime $\mathfrak{P}$ whose norm we can calculate, we note that we may include the term $C(\mathfrak{P}, y)$ (or $B(\mathfrak{P}, y)$ under GRH) in the inequalities above, resulting in larger discriminant bounds. This was done by Selmane [9], who produced tables giving bounds on discriminants of number fields containing a single prime of a given norm. Unfortunately, these tables do not suffice for our purposes. In several of our results we required bounds derived using knowledge of several primes of K, and hence including several local corrections. Our discriminant bounds then take the following form:

Theorem 2.4. *Let $K/\mathbb{Q}$ be a number field of degree n with r_1 real places, let y be a positive real number, and let S be a finite set of primes of K of known norms. Set $f(x) = \left(\frac{3}{x^3}(\sin x - x\cos x)\right)^2$ and*

$$G(x) = \begin{cases} (1-|x|)\cos|\pi x| + \frac{1}{\pi}\sin|\pi x|, & \text{if } x \in [-1, 1] \\ 0, & \text{otherwise.} \end{cases}$$

(1) If we do not assume GRH, then for all $y > 0$,

$$\log|d_K| \geq r_1 + n(\gamma + \log 4\pi) - \frac{12\pi}{5\sqrt{y}} - I(y) + \sum_{\mathfrak{P}\in S} C(\mathfrak{P}, y),$$

where $I(y)$ and $C(\mathfrak{P}, y)$ are given in terms of $f(x)$ by (2.1) and (2.2), respectively.

(2) Under the assumption of the GRH, for all $y > 0$,

$$\log|d_k| \geq r_1\frac{\pi}{2} + n(\gamma + \log 8\pi) - J(y) + \sum_{\mathfrak{P}\in S} B(\mathfrak{P}, y),$$

where $J(y)$ and $B(\mathfrak{P}, y)$ are given in terms of $G(x)$ by (2.3) and (2.4), respectively.

To obtain the best possible discriminant bound we take y so that the right-hand side of the appropriate inequality above is as large as possible.

Table 1. Degree 10 discriminant bounds

S	$r_1 = 0$	$r_1 > 0$
$\emptyset$	1.569 E08	5.595 E08
$\{2\}$	5.672 E08	2.136 E09
$\{2,2\}$	2.189 E09	8.641 E09
$\{2,2,2\}$	8.935 E09	3.673 E10
$\{3\}$	4.032 E08	1.517 E09
$\{3,3\}$	1.105 E09	4.360 E09
$\{3,3,3\}$	3.202 E09	1.315 E10
$\{5\}$	2.764 E08	1.038 E09
$\{5,5\}$	5.175 E08	2.036 E09
$\{5,5,5\}$	1.021 E09	4.183 E09
$\{7\}$	2.253 E08	8.437 E08
$\{7,7\}$	3.410 E08	1.335 E09
$\{7,7,7\}$	5.417 E08	2.205 E09

Table 2. Degree 12 discriminant bounds

S	$r_1 = 0$	$r_1 > 0$
$\emptyset$	2.753 E10	1.057 E11
$\{2\}$	1.087 E11	4.359 E11
$\{2,2\}$	4.521 E11	1.883 E12
$\{2,2,2\}$	1.966 E12	8.466 E12
$\{3\}$	7.724 E10	3.096 E11
$\{3,3\}$	2.280 E11	9.496 E11
$\{3,3,3\}$	7.040 E11	3.030 E12
$\{5\}$	5.281 E10	2.115 E11
$\{5,5\}$	1.063 E11	4.420 E11
$\{5,5,5\}$	2.234 E11	9.599 E11
$\{7\}$	4.281 E10	1.710 E11
$\{7,7\}$	6.954 E10	2.880 E11
$\{7,7,7\}$	1.174 E11	5.024 E11

2.4. *Tables*

In this section, we provide tables of lower bounds on $|d_K|$ for number fields K of degree $n = 2m$ where $5 \leq m \leq 9$. The set S contributing to local corrections is described by the norms of the prime ideals. For example, line 3 of Table 1 gives the discriminant bounds for a number field K of degree 10 over $\mathbb{Q}$ containing two primes of norm 2. We should point out that the first line with set $S = \emptyset$ of each table will give the same lower bound that Diaz y Diaz [3] calculated over twenty-five years ago without local corrections.

Our calculations were performed by Maple using Theorem 2.4. Our results are reported to four significant figures because this is sufficient for our work. In Section 4, we will either know that our number field K is totally complex or not totally complex. Hence we only provide bounds for $r_1 = 0$ and $r_1 > 0$. Note that these bounds could be calculated to full integer accuracy, for odd degrees, and for all signatures.

3. Upper bounds on discriminants

3.1. *Different and discriminant*

Our goal in this section is to obtain useful upper bounds on discriminants. We begin by recalling formulae relating differents and discriminants in towers of field extensions.

Proposition 3.1. *Let K/F and $F/\mathbb{Q}$ be number field extensions. Then the different $\mathfrak{D}_{K/F}$ and the discriminants $\Delta_{K/F}$, $\Delta_{K/\mathbb{Q}}$, and $\Delta_{F/\mathbb{Q}}$ satisfy the following properties.*

Table 3. Degree 14 discriminant bounds

S	$r_1 = 0$	$r_1 > 0$
$\emptyset$	5.440 E12	2.213 E13
$\{2\}$	2.300 E13	9.700 E13
$\{2,2\}$	1.014 E14	4.415 E14
$\{2,2,2\}$	4.644 E14	2.077 E15
$\{3\}$	1.634 E13	6.888 E13
$\{3,3\}$	5.117 E13	2.225 E14
$\{3,3,3\}$	1.662 E14	7.433 E14
$\{5\}$	1.115 E13	4.698 E13
$\{5,5\}$	2.379 E13	1.033 E14
$\{5,5,5\}$	5.258 E13	2.347 E14
$\{7\}$	9.011 E12	3.789 E13
$\{7,7\}$	1.547 E13	6.706 E13
$\{7,7,7\}$	2.747 E13	1.222 E14

Table 4. Degree 16 Discriminant bounds

S	$r_1 = 0$	$r_1 > 0$
$\emptyset$	1.177 E15	5.019 E15
$\{2\}$	5.261 E15	2.310 E16
$\{2,2\}$	2.434 E16	1.098 E17
$\{2,2,2\}$	1.162 E17	5.367 E17
$\{3\}$	3.735 E15	1.640 E16
$\{3,3\}$	1.227 E16	5.534 E16
$\{3,3,3\}$	4.156 E16	1.919 E17
$\{5\}$	2.546 E15	1.117 E16
$\{5,5\}$	5.695 E15	2.565 E16
$\{5,5,5\}$	1.311 E16	6.049 E16
$\{7\}$	2.052 E15	8.999 E15
$\{7,7\}$	3.690 E15	1.659 E16
$\{7,7,7\}$	6.821 E15	3.138 E16

Table 5. Degree 18 discriminant bounds

S	$r_1 = 0$	$r_1 > 0$
$\emptyset$	2.738 E17	1.213 E18
$\{2\}$	1.280 E18	5.820 E18
$\{2,2\}$	6.166 E18	2.867 E19
$\{2,2,2\}$	3.049 E19	1.447 E20
$\{3\}$	9.090 E17	4.131 E18
$\{3,3\}$	3.107 E18	1.444 E19
$\{3,3,3\}$	1.090 E19	5.174 E19
$\{5\}$	6.191 E17	2.812 E18
$\{5,5\}$	1.439 E18	6.689 E18
$\{5,5,5\}$	3.434 E18	1.628 E19
$\{7\}$	4.981 E17	2.261 E18
$\{7,7\}$	9.303 E17	4.315 E18
$\{7,7,7\}$	1.779 E18	8.419 E18

Table 6. Degree 12 bounds with GRH

S	$r_1 = 0$	$r_1 > 0$
$\emptyset$	3.727 E10	1.534 E11
$\{2\}$	1.608 E11	6.992 E11
$\{2,2\}$	7.398 E11	3.377 E12
$\{2,2,2\}$	3.604 E12	1.719 E13
$\{3\}$	1.142 E11	4.968 E11
$\{3,3\}$	3.734 E11	1.703 E12
$\{3,3,3\}$	1.291 E12	6.153 E12
$\{5\}$	7.798 E10	3.385 E11
$\{5,5\}$	1.733 E11	7.898 E11
$\{5,5,5\}$	4.073 E11	1.937 E12
$\{7\}$	6.290 E10	2.723 E11
$\{7,7\}$	1.120 E11	5.080 E11
$\{7,7,7\}$	2.099 E11	9.929 E11

(1) The different $\mathfrak{D}_{K/F}$ is an ideal of K divisible only by ideals of K which are ramified in K/F.
(2) The discriminant $\Delta_{K/F}$ is an ideal of F divisible only by ideals of F which ramify in K/F.
(3) The norm $N_F^K(\mathfrak{D}_{K/F}) = \Delta_{K/F}$.
(4) The discriminant $\Delta_{K/\mathbb{Q}} = \Delta_{F/\mathbb{Q}}^{[K:F]} N_{\mathbb{Q}}^F(\Delta_{K/F})$.

Proof. These are well known properties, and may be found in standard texts, such

Table 7. Upper bounds on exponents of the different $\mathfrak{D}_{K/L}$

prime power	p =2	p =3	p =5	p =7
$\mathfrak{P}^1$	0	0	0	0
$\mathfrak{P}^2$	5	1	1	1
$\mathfrak{P}^3$	2	8	2	2
$\mathfrak{P}^4$	19	3	3	3
$\mathfrak{P}^5$	4	4	14	4
$\mathfrak{P}^6$	17	17	5	5
$\mathfrak{P}^7$	6	6	6	20
$\mathfrak{P}^8$	55	7	7	7
$\mathfrak{P}^9$	8	44	8	8

as [10, Chapter III], [2, Chapter 2] or [6, Section 4.2]. □

For any field F, we note that the absolute discriminant d_F is one of the two generators of the ideal $\Delta_{F/\mathbb{Q}}$. We also recall the following bound on the different.

Proposition 3.2. *[10, pg. 58] Let K/F be an extension of number fields, and let $\mathfrak{P}/\mathfrak{p}$ be a prime with ramification index e. Then the exponent of $\mathfrak{P}$ in the different $\mathfrak{D}_{K/F}$ of K/F is equal to*

$$e - 1 + W(\mathfrak{P}),$$

where $W(\mathfrak{P})$ is a nonnegative integer which is positive exactly when $e \in \mathfrak{P}$. When $e \in \mathfrak{P}$, we have the inequality

$$1 \leq W(\mathfrak{P}) \leq v_{\mathfrak{P}}(e).$$

3.2. *Extensions of quadratic fields*

Let F be a specified quadratic field ramified only at p, in which the prime ideal (p) factors as $\mathfrak{p}^2$, and let K be a degree m extension of F, so that $n = 2m$. We will assume that K/F is ramified only at $\mathfrak{p}$. The discriminant of F, d_F, is known, and we wish to get an upper bound on d_K.

In the extension K/F, let $\mathfrak{P}$ be a prime lying over $\mathfrak{p}$, with ramification index e. Using Proposition 3.2, and the fact that the ramification index of $\mathfrak{P}/p$ is $2e$ we find that the power of $\mathfrak{P}$ dividing $\mathfrak{D}_{K/F}$ is

$$v_{\mathfrak{P}}(\mathfrak{D}_{K/F}) \leq e - 1 + v_{\mathfrak{P}}(e) = e - 1 + 2ev_p(e),$$

where v_p is the p-adic valuation. Table 7 gives values for this bound for small values of p and ramification indices up to $e = 9$.

We now factor

$$\mathfrak{p}\mathfrak{O}_K = \prod_{i=1}^{g} \mathfrak{P}_i^{e_i},$$

where each prime $\mathfrak{P}_i$ has inertial degree f_i. Summing over each prime of K lying above $\mathfrak{p}$ and taking the norm from K to F, we have the following theorem:

Proposition 3.3. *The discriminant $\Delta_{K/F}$ satisfies*

$$v_{\mathfrak{p}}(\Delta_{K/F}) \leq \sum_{i=1}^{g} f_i(e_i - 1 + 2e_i v_p(e_i)).$$

Combining this bound with Proposition 3.1, we bound $\Delta_{K/\mathbb{Q}}$.

Corollary 3.4. *Let F be a quadratic field ramified only at p, and let K be an extension of F of degree m, ramified only at the prime above p. Set*

$$M = \sum_{i=1}^{g} f_i(e_i - 1 + 2e_i v_p(e_i)).$$

Then

$$v_p(d_K) = v_p(\Delta_{K/\mathbb{Q}}) \leq m v_p(d_F) + M.$$

Since $|d_K|$ must be a power of p, we see that

$$|d_K| \leq |d_F|^m p^M.$$

4. Nonexistence results

We now use these bounds to prove the nonexistence of certain extensions of quadratic fields. Our strategy will be to show that the upper bound on $|d_K|$ derived from the different is smaller than the lower bound on $|d_K|$ derived from the analytic techniques of Section 2.

Theorem 4.1. *There are no extensions of $\mathbb{Q}(i)$ of degree 5, 6, or 7, ramified only at the prime above 2.*

Proof. Let $F = \mathbb{Q}(i)$, and let $\mathfrak{p}$ be the unique prime of F lying over 2.

For a degree 5 extension K of F, one sees easily that K is totally complex since F is totally complex. Also, the largest possible value of $v_{\mathfrak{p}}(\Delta_{K/F})$ is 19, occurring when $\mathfrak{p}\mathfrak{O}_K = \mathfrak{P}_1^4\mathfrak{P}_2$, where both $\mathfrak{P}_i$ have inertial degree 1. Hence

$$v_2(\Delta_{K/\mathbb{Q}}) \leq 5v_2(\Delta_{F/\mathbb{Q}}) + 19 = 29.$$

Hence $|d_K| \leq 2^{29} \approx 5.369 \times 10^8$. However, for a field with at least one prime of norm 2, Table 1 indicates that the discriminant must be greater than 5.672×10^8. Hence, if K exists, it cannot contain a prime of norm 2. However, if K contains no prime of norm 2, it is easy to see that K/F must be unramified, so that $d_K =$

$(d_F)^5 = 2^{10} = 1024$, which is far less than the unconditional bound of 1.569×10^8 (see Table 1) for any totally complex number field of degree 10. Hence K cannot exist.

If K/F is of degree 6, and is ramified only at $\mathfrak{p}$, we see easily that if we allow one or more ideals of norm 2 in K, then

$$v_2(\Delta_{K/\mathbb{Q}}) \leq 6v_2(\Delta_{F/\mathbb{Q}}) + 19 + 5 = 36$$

(with the maximum occurring when $\mathfrak{p}\mathfrak{O}_K = \mathfrak{P}_1^4\mathfrak{P}_2^2$) so that $|d_K| \leq 2^{36} \approx 6.872 \times 10^{10}$. This is significantly smaller than 1.087×10^{11}, which (from Table 2) is the smallest possible value for the discriminant of a totally complex degree 12 number field containing one or more primes of norm 2. Hence, any such K must not contain a prime of norm 2. However, if K contains no primes of norm 2, one sees easily that the largest possible discriminant is $2^{27} \approx 1.343 \times 10^8$ (occurring when $\mathfrak{p}\mathfrak{O}_K = \mathfrak{P}_1^2$ with $f_1 = 3$) is much smaller than the unconditional bound of 2.753×10^{10} (see Table 2) for any totally complex number field of degree 12.

For a degree 7 extension K/F ramified only at p, we see again that K is totally complex. If we allow one or more prime ideals of norm 2, we see that $|d_K| \leq 4^7 2^{24} = 2^{38} \approx 2.749 \times 10^{11}$, occurring when $\mathfrak{p}\mathfrak{O}_K = \mathfrak{P}_1^4\mathfrak{P}_2^2\mathfrak{P}_3$ is a product of three ideals of norm 2. Since this largest possible value is smaller than the lower bound of 5.440×10^{12} (see Table 3) for any totally complex number field of degree 14 (regardless of what ideals it contains), we see that K/F can not exist. □

Theorem 4.2. *There are no extensions of* $\mathbb{Q}(\sqrt{2})$ *of degree 7 ramified only at the prime above 2.*

Proof. Let $F = \mathbb{Q}(\sqrt{2})$, and let $\mathfrak{p}$ be the unique prime of F lying over 2. As in the proof of the degree 7 part of Theorem 4.1, we see that for K/F of degree 7, $|d_K| \leq 8^7 2^{24} = 2^{45} \approx 3.519 \times 10^{13}$. Since F is real and K/F is odd, we know that K/F will have at least one real place. Examining Table 3, we see that for a degree 14 field K with any real places and at least one prime of norm 2, $|d_K| \geq 9.700 \times 10^{13}$. Thus, if K exists, it cannot contain any ideals of norm 2. However, if we restrict the ramification so that K contains no ideals of norm 2, we find that $|d_K| \leq 8^7 2^{10} = 2^{31} \approx 2.148 \times 10^9$, (with the maximum occurring for $\mathfrak{p}\mathfrak{O}_K = \mathfrak{P}_1^2\mathfrak{P}_2$ with $f_1 = 2$ and $f_2 = 3$), which is far less than the lower bound of 2.213×10^{13} for fields of degree 14 with at least one real place. Hence K does not exist. □

Theorem 4.3. *There are no extensions of* $\mathbb{Q}(\sqrt{-2})$ *of degree 7 ramified only at the prime above 2.*

Proof. Let $F = \mathbb{Q}(\sqrt{-2})$, and let $\mathfrak{p}$ be the unique prime above 2 in F. Exactly as in Theorem 4.2, we see that $|d_K| \leq 2^{45} \approx 3.519 \times 10^{13}$, and this bound arises from a factorization having three ideals of norm 2. In this case, K must be totally complex and the bound for a degree 14 totally complex field containing at least two ideals of norm 2 (from Table 3) is 1.014×10^{14}. So if K exists, it must have 0 or 1 ideals of norm 2. Restricting K to have at most one ideal of norm 2, we find that $|d_K| \leq 8^7 2^{19} = 2^{40} \approx 1.100 \times 10^{12}$, with the highest value occurring for the factorization $\mathfrak{p}\mathfrak{O}_K = \mathfrak{P}_1^4 \mathfrak{P}_2$ with $f_1 = 1$ and $f_2 = 3$. This is lower than the lower bound of 5.440×10^{12} for arbitrary totally complex fields of degree 14, so K can not exist. $\square$

Theorem 4.4. *There are no extensions of $F = \mathbb{Q}(\sqrt{-3})$ of degree 5,7,8 ramified only at the prime above 3. If we assume the GRH, then there are no extensions of F of degree 6 ramified only at the prime above 3.*

Proof. Let $F = \mathbb{Q}(\sqrt{-3})$ and let $\mathfrak{p}$ be the unique prime of F lying over 3.

Suppose K/F is of degree 5. From Table 7, we see that $v_{\mathfrak{p}}(\Delta_{K/F}) \leq 9$ (with the highest possible value occurring when $\mathfrak{p}\mathfrak{O}_K = \mathfrak{P}_1^3 \mathfrak{P}_2^2$), so that $|d_K| \leq 3^5 3^9 = 3^{14} \approx 4.783 \times 10^6$. Since a totally complex degree 10 extension of $\mathbb{Q}$ must have discriminant at least 1.569×10^8, we see that K can not exist.

If K/F is of degree 7, we see that $v_{\mathfrak{p}}(\Delta_{K/F}) \leq 17$, with the highest value occurring when $\mathfrak{p}\mathfrak{O}_K = \mathfrak{P}_1^6 \mathfrak{P}_2$. Then $|d_K| \leq 3^7 3^{17} = 3^{24} \approx 2.825 \times 10^{11}$, but any degree 14 extension of $\mathbb{Q}$ must have discriminant larger than 5.440×10^{12}. Hence K can not exist.

If K/F is of degree 8, then $v_{\mathfrak{p}}(\Delta_{K/F}) \leq 18$, so that $|d_K| \leq 3^8 3^{18} = 3^{26} \approx 2.542 \times 10^{12}$. Since a degree 16 extension of $\mathbb{Q}$ must have discriminant at least 1.177×10^{15}, we see that K does not exist.

If K/F is of degree 6, we see easily that if we allow one or more ideals of norm 3 in K, then $v_{\mathfrak{p}}(\Delta_{K/F}) \leq 17$, so that $|d_K| \leq 3^6 3^{17} = 3^{23} \approx 9.415 \times 10^{10}$. Unfortunately, this is larger than 7.724×10^{10}, which (from Table 2) is the smallest possible value for the discriminant of a totally complex degree 12 number field containing a prime of norm 3. Therefore we must assume GRH to get a better lower bound. Table 6 indicates that the GRH lower bound for the discriminant of a degree 12 totally complex field with a single prime of norm 3 is 1.142×10^{11}. Since $9.415 \times 10^{10} < 1.142 \times 10^{11}$, K does not exist. If we restrict to the case where K contains no prime of norm 3, then $|d_K| \leq 3^{22} \approx 3.139 \times 10^{10}$. This is smaller than the GRH lower bound of 3.727×10^{10} for the discriminant of a number field of degree 12 over $\mathbb{Q}$. (Note that by using a local correction with an ideal of norm 9, the case with no ideal of norm 3 could have been done unconditionally.) $\square$

Theorem 4.5. *There are no extensions of $F = \mathbb{Q}(\sqrt{5})$ of degree 9 ramified only at the prime above 5.*

Proof. Let $F = \mathbb{Q}(\sqrt{5})$ and let $\mathfrak{p}$ be the unique prime of F lying over 5. If K/F is of degree 9 and is ramified only at $\mathfrak{p}$ then the maximum discriminant arises in the case $\mathfrak{p} = \mathfrak{P}_1^5 \mathfrak{P}_2^4$. We have

$$v_p(\Delta_{K/\mathbb{Q}}) \leq 9 v_p(\Delta_{F/\mathbb{Q}}) + 17 = 26.$$

So $|d_K| \leq 5^{26}$ which is approximately 1.491×10^{18}. The lower bound on a degree 18 field with at least one real place and an ideal of norm 5 is 2.812×10^{18} which gives a contradiction. If K/F is of degree 9 and has no primes of norm 5, then $1.213 \times 10^{18} \leq |d_K| \leq 5^{15} \approx 3.052 \times 10^{10}$ which also gives a contradiction. □

Theorem 4.6. *There are no extensions of $F = \mathbb{Q}(\sqrt{-7})$ of degree 5 or 6 ramified only at the prime above 7.*

Proof. Let $F = \mathbb{Q}(\sqrt{-7})$. If K is a degree 5 or 6 extension of F, then the extension K/F can not be wildly ramified at the prime above 7. Hence the extension $K/\mathbb{Q}$ is tamely ramified at 7 and $|d_K| \leq 7^9$ or 7^{11} respectively. These values are easily less than the respective lower bounds of 1.569×10^8 and 2.753×10^{10}. □

Finally, recall that every Galois extension with group embedding in S_n must arise from a degree n extension. Hence each nonexistence theorem for degree n implies the nonexistence of certain nonsolvable Galois extensions of the quadratic field F. For example, Theorem 4.1 implies there are no extensions of $F = \mathbb{Q}(i)$ with Galois group A_5, S_5, A_6, S_6, $PSL_2(\mathbb{F}_7)$, A_7, or S_7 which are ramified only above the prime 2.

5. Addendum

After the submission of this paper, John Jones informed the authors that he has independently obtained the nonexistence results derived in Theorems 4.1, 4.2, and 4.3 for fields ramified only at two. His work [5] uses techniques which differ significantly from those presented here.

References

[1] Sharon Brueggeman. Septic number fields which are ramified only at one small prime. *J. Symbolic Comput.*, 31(5):549–555, 2001.
[2] Henri Cohen. *Advanced topics in computational number theory*, volume 193 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2000.
[3] Francisco Diaz y Diaz. *Tables minorant la racine n-ième du discriminant d'un corps de degré n*, volume 6 of *Publications Mathématiques d'Orsay 80 [Mathematical Publications of Orsay 80]*. Université de Paris-Sud Département de Mathématique, Orsay, 1980.
[4] John Jones. Tables of number fields with prescribed ramification. http://math.la.asu.edu/~jj/numberfields.
[5] John Jones. Number fields unramified away from 2. *Preprint*, 2006.
[6] Władysław Narkiewicz. *Elementary and analytic theory of algebraic numbers*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, third edition, 2004.
[7] A. M. Odlyzko. Bounds for discriminants and related estimates for class numbers, regulators and zeros of zeta functions: a survey of recent results. *Sém. Théor. Nombres Bordeaux (2)*, 2(1):119–141, 1990.
[8] Georges Poitou. Sur les petits discriminants. In *Séminaire Delange-Pisot-Poitou, 18e année: (1976/77), Théorie des nombres, Fasc. 1 (French)*, pages Exp. No. 6, 18. Secrétariat Math., Paris, 1977.

[9] Schehrazad Selmane. Odlyzko-Poitou-Serre lower bounds for discriminants for some number fields. *Maghreb Math. Rev.*, 8(1-2):151–162, 1999.
[10] Jean-Pierre Serre. *Local fields*, volume 67 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1979. Translated from the French by Marvin Jay Greenberg.
[11] Lesseni Sylla. The nonexistence of nonsolvable octic number fields ramified only at one small prime. *Math. Comp.*, 2006. to appear.
[12] Lessini Sylla. Nonsolvable nonic number fields ramified only at one small prime. *Journal de Thorie des Nombres de Bordeaux*, 2005. to appear.