Three-dimensional Galois Representations with Conjectural Connections to Arithmetic Cohomology

Darrin Doud[1]

1 Introduction

In [4], Ash and Sinnott conjecture that any Galois representation having niveau 1 which satisfies a certain parity condition is attached in a specific way to a Hecke eigenclass in cohomology, and they make a prediction about exactly where the relevant cohomology class should lie. They give examples of reducible three-dimensional representations which appear to be attached to cohomology eigenclasses, in the sense that the characteristic polynomials of Frobenius elements for small primes correspond exactly to the Hecke eigenvalues of certain eigenclasses. The question of proving this connection for all primes seems to be difficult; however, in [6] Ash and Tiep develop techniques for proving that certain irreducible three-dimensional symmetric square representations are in fact attached to cohomology classes. Until recently there were no known examples of three-dimensional irreducible non-symmetric square characteristic p Galois representations which seem to be attached to cohomology eigenclasses. In this paper we extend the original conjecture of Ash and Sinnott to include irreducible niveau 2 representations and give an example of a niveau 2 Galois representation which is neither reducible nor obtained as the symmetric square of a two dimensional representation, but for which the conjectured connection with arithmetic cohomology appears to hold, at least for prime $\ell \leq 47$. We also briefly discuss the computational techniques needed to demonstrate the apparent connection.

We note that the forthcoming paper [3] of the author, Avner Ash and David Pollack extends the conjecture given here to include reducible (but semisimple) representations, deals with higher dimensional and higher niveau representations, and includes many more computational verifications of the conjecture.

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2 Conjectural Connections Between Galois Representations and Cohomology

Following the discussion in [4], let $\Gamma = \mathrm{SL}_3(\mathbb{Z})$, and let $\Gamma_0(N)$ be the subgroup of Γ consisting of all matrices having first row congruent to $(*, 0, \ldots, 0)$ modulo N. Let S_N be the subsemigroup of $\mathrm{GL}_3(\mathbb{Q})$ having integral entries, positive determinant prime to N, and first row congruent to $(*, 0, \ldots, 0)$ modulo N. Then $(\Gamma_0(N), S_N)$ is a congruence Hecke pair of level N in the sense of [2]. We define the Hecke algebra $\mathcal{H}(N)$ to be the $\bar{\mathbb{F}}_p$-algebra of double cosets $\Gamma_0(N)S_N\Gamma_0(N)$, and note that it acts on cohomology or homology with coefficients in any $\mathbb{F}S_N$-module, as described in [5] or [1]. We call an element of $\mathcal{H}(N)$ a Hecke operator when it acts on homology or cohomology. For $\ell \nmid N$, $\mathcal{H}(N)$ contains the double cosets $\Gamma_0(N)D(\ell, k)\Gamma_0(N)$ where $D(\ell, k)$ is the diagonal matrix

$$\begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & \ell & & \\ & & & & \ddots & \\ & & & & & \ell \end{pmatrix}$$

with ℓ on the diagonal in the last k positions, and we denote the corresponding Hecke operator by $T(\ell, k)$.

Definition 2.1. If V is an $\mathcal{H}(pN)$-module, and $v \in V$ is an eigenvector of all the $T(\ell, k)$ with $\ell \nmid pN$, such that $T(\ell, k)v = a(\ell, k)v$, with $a(\ell, k) \in \bar{\mathbb{F}}_p$, and $\rho : G_{\mathbb{Q}} \to \mathrm{GL}_3(\bar{\mathbb{F}}_p)$ is a Galois representation unramified outside pN, then we say that ρ is attached to v if

$$\sum_{k=0}^{3} (-1)^k \ell^{k(k-1)/2} a(\ell, k) X^k = \det(I - \rho(\mathrm{Fr}_\ell)X)$$

for all $l \nmid pN$.

Given ρ satisfying certain conditions, our conjecture will predict the existence of an eigenclass v with ρ attached. The main difficulty is specifying exactly which module will contain the eigenclass.

2.1 Defining the Level and Nebentype

We define the level N and nebentype ϵ of a Galois representation $\rho : G_{\mathbb{Q}} \to \mathrm{GL}_n(\bar{\mathbb{F}}_p)$ exactly as in [4], in a straightforward generalization of Serre's definitions of the level and nebentype of a two dimensional representation in [8]. Let G_q be a chosen decomposition group above the prime q in $G_{\mathbb{Q}}$, and let $G_{q,0} \supseteq G_{q,1} \supseteq \dots$ be the filtration of ramification subgroups. In particular, $G_{q,0} = I_q$ is an inertia group at q. Let $M = \bar{\mathbb{F}}_p^n$ be acted on by $G_{\mathbb{Q}}$ via ρ.

Definition 2.2. Let

$$n_q = \sum_{i=0}^{\infty} \frac{1}{[\rho(G_{q,0}) : \rho(G_{q,i})]} \dim M/M^{G_{q,i}}.$$

Then we define

$$N(\rho) = \prod_{q \neq p} q^{n_q}.$$

We note that both the sum defining n_q and the product defining N are in fact finite, just as is the case in [8].

In order to define the nebentype of ρ, we factor the determinant

$$\det \rho = \omega^k \epsilon,$$

where ω is the cyclotomic character modulo p, and ϵ is unramified at p. We may then consider ϵ as a Dirichlet character modulo $N = N(\rho)$

$$\epsilon : (\mathbb{Z}/N\mathbb{Z})^* \to \bar{\mathbb{F}}_p^*.$$

We use this character to define a character

$$\epsilon = \epsilon(\rho) : S_N \to (\mathbb{Z}/N\mathbb{Z})^* \to \bar{\mathbb{F}}_p^*,$$

where the first map is projection onto the $(1,1)$ element of a matrix in S_N and the second is the Dirichlet character defined above.

We now take

$$V(\epsilon) = V \otimes \bar{\mathbb{F}}_p,$$

and note that $V(\epsilon)$ is both a $\Gamma_0(N)$-module and an S_{pN}-module, with the action on V given by reduction modulo p, and the action on $\mathbb{F}_p$ given by ϵ. Hence we may compute the cohomology of $\Gamma_0(N)$ with coefficients in $V(\epsilon)$, and that cohomology is an $\mathcal{H}(pN)$-module.

2.2 Irreducible $\mathrm{GL}_n(\mathbb{F}_p)$ Modules

Definition 2.3. An n-tuple $(a_1, \dots, a_n)$ of integers is said to be p-restricted if for all $i < n$,

$$0 \le a_i - a_{i+1} \le p - 1 \quad \text{and} \quad 0 \le a_n < p - 1.$$

The following theorem is well known ([7]).

Theorem 2.4. *The set of irreducible $GL_n(\mathbb{F}_p)$-modules is in one to one correspondence with the set of p-restricted n-tuples.*

In fact, we may describe the irreducible $GL_n(\mathbb{F}_p)$-module associated to the n-tuple $(a_1, \ldots, a_n)$ as the unique simple submodule of the dual Weyl module with highest weight $(a_1, \ldots, a_n)$, and we will denote this module by $F(a_1, \ldots, a_n)$.

Given any n-tuple $(b_1, \ldots, b_n)$ of integers, we will use the notation $(b_1, \ldots, b_n)'$ to denote an n-tuple $(a_1, \ldots, a_n)$ which is p-restricted, and such that each $a_i \equiv b_i \pmod{p-1}$. Such an n-tuple may not be uniquely defined—if it is not, then we interpret statements about $(b_1, \ldots, b_n)'$ to be true if they are true for some choice of $(a_1, \ldots, a_n)$ as above.

2.3 Main Conjecture

Let ω be the cyclotomic character modulo p, and let $\psi, \psi' : G_{p,0} \to \mathbb{F}_{p^2}$ be the fundamental characters of niveau 2 (so that in particular, $\psi' = \psi^p$ and ψ has order $p^2 - 1$).

Conjecture 2.5. *Let $\rho : G_{\mathbb{Q}} \to GL_3(\bar{\mathbb{F}}_p)$ be a continuous irreducible Galois representation, which takes complex conjugation to a nonscalar matrix. If we have*

$$\rho|_{I_p} \sim \begin{pmatrix} \varphi_1 & * & * \\ & \varphi_2 & * \\ & & \varphi_3 \end{pmatrix},$$

then,

1. *if $\varphi_i = \omega^{\alpha_i}$, we set*
$$a_i = \alpha_i;$$
2. *if $\varphi_i = \psi^m$, and $\varphi_j = \psi'^m$ (with $i < j$), we write $m = a + bp$, with $0 \le a - b \le p - 1$, and take*
$$a_i = a, \quad a_j = b;$$

Letting $N = N(\rho)$, $\epsilon = \epsilon(\rho)$, and

$$V = F(a_1 - 2, a_2 - 1, a_3 - 0)',$$

we have that ρ is attached to a cohomology eigenclass in

$$H^3(\Gamma_0(N), V(\epsilon)).$$

Remark 2.6. Note that as stated, the conjecture does not deal with reducible Galois representations or with Galois representations which have niveau 3. A generalization which does deal with these cases will appear in [3]. In the niveau 1 case, Conjecture 2.5 reduces to the irreducible case of Conjecture 2.2 in [4].

Remark 2.7. We note that in the case of a tamely ramified Galois representation there are six permutations of the diagonal characters. In the generic case (where no two permutations give the same predicted weight) we thus expect to find an eigenclass attached to the Galois representation in each of at least six different weights. This is analogous to the existence of "companion forms" in Serre's conjecture. Note that in the wildly ramified case, the wild ramification restricts the allowed permutations on the diagonal, so that the existence of companion forms is not predicted, except as the wild ramification may permit.

3 A Galois Representation with Niveau 2

Let K be the splitting field of the polynomial $x^3 - 10x - 15$. Then the Galois group of K over $\mathbb{Q}$ is isomorphic to S_3, and the only primes which ramify in K are 5 (with ramification index 3) and 83 (with ramification index 2). Using GP/PARI, we may compute the ideal class group H of K: it is isomorphic to $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$. Then if L is the Hilbert class field of K, we know that $L/\mathbb{Q}$ is Galois, $\mathrm{Gal}(L/K) \cong H$, and that the action of $\mathrm{Gal}(K/\mathbb{Q})$ on $\mathrm{Gal}(L/K)$ by conjugation is the same as the action of $\mathrm{Gal}(K/\mathbb{Q})$ on H.

This last action may be explicitly computed–since K is only degree 6 over $\mathbb{Q}$, the computation takes only seconds using GP/PARI. We find that (relative to a suitable basis) this action is given by

$$(1,2,3) \mapsto \begin{pmatrix} 2 & 2 \\ 1 & 0 \end{pmatrix}$$

and

$$(1,2) \mapsto \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}.$$

We then have an exact sequence

$$0 \to H \to \mathrm{Gal}(L/\mathbb{Q}) \to S_3 \to 0,$$

where the action of S_3 on H is given above. Such exact sequences are parameterized by $H^2(S_3, H)$. This cohomology group may be explicitly calculated using Magma: we find that it is trivial, so that the sequence

Class	1	2	3	4	5	6	7	8	9	10
Size	1	9	1	1	6	6	6	6	9	9
Order	1	2	3	3	3	3	3	3	6	6
χ_1	1	1	1	1	1	1	1	1	1	1
χ_2	1	-1	1	1	1	1	1	1	-1	-1
χ_3	2	0	2	2	2	-1	-1	-1	0	0
χ_4	2	0	2	2	-1	2	-1	-1	0	0
χ_5	2	0	2	2	-1	-1	-1	2	0	0
χ_6	2	0	2	2	-1	-1	2	-1	0	0
χ_7	3	-1	3ζ	$3\zeta^2$	0	0	0	0	$-\zeta$	$-\zeta^2$
χ_8	3	1	3ζ	$3\zeta^2$	0	0	0	0	ζ	ζ^2
χ_9	3	-1	$3\zeta^2$	3ζ	0	0	0	0	$-\zeta^2$	$-\zeta$
χ_{10}	3	1	$3\zeta^2$	3ζ	0	0	0	0	ζ^2	ζ

Table 1. Character Table of G

must split. Alternatively, we could appeal to the main theorem of [9], which proves directly that the sequence splits.

We now know that $G = \mathrm{Gal}(L/\mathbb{Q})$ is the semidirect product of S_3 and H. The character table of G (as computed by Magma) is given in Table 1, where ζ denotes a cube root of unity. Note that the character table over $\bar{\mathbb{F}}_5$ is the same as the character table over $\mathbb{C}$, since G has order 54, which is relatively prime to 5.

We now define ρ_i (for $1 \le i \le 4$) to be a three-dimensional representation of G defined over $\bar{\mathbb{F}}_5$ corresponding to the character χ_{i+6}. We will also denote by ρ_i the Galois representation obtained as the composition

$$G_{\mathbb{Q}} \to G \xrightarrow[\rho_i]{} \mathrm{GL}_3(\bar{\mathbb{F}}_5),$$

where the first map is the projection $G_{\mathbb{Q}} \to \mathrm{Gal}(L/\mathbb{Q}) = G$. Thus, each Galois representation ρ_i has image isomorphic to G, and kernel equal to G_L. Note that all of these representations map complex conjugation to a nonscalar matrix, so they are within the purview of our conjecture.

Using GP/PARI, we may compute the traces of Frobenius elements under the map ρ_2 as follows. First, note that all the primes $\ell \le 47$ with inertial degree 3 in L (except for the prime 3) already have inertial degree 3 in K, so their Frobenius in L is noncentral of order 3. The primes above

ℓ	2	3	7	11	13	17	19	23	29	31	37	41	43	47
Order(Fr_ℓ)	6	3	3	3	6	3	6	3	3	3	3	3	6	6
$\mathrm{Tr}(\rho_2(\mathrm{Fr}_\ell))$	ζ	0	0	0	ζ	0	ζ	0	0	0	0	0	ζ^2	ζ^2

Table 2. Orders and Traces of Frobenius for ρ_2

3 in K, on the other hand, have inertial degree 1. One checks that they have order 3 in the ideal class group of K (using the GP/PARI command `bnfisprincipal`), and that they are not fixed by the action of S_3, so the Frobenius at 3 is again a noncentral element of order 3. Finally, the conjugacy classes of the Frobenius elements of order 6 may be determined by applying `bnfisprincipal` to the primes with inertial degree 2 in $K/\mathbb{Q}$. In this fashion, we determine that the Frobenius elements above 2, 13, and 19 are in one conjugacy class, and that the Frobenius elements above 43 and 47 are in the other conjugacy class. We choose ζ to be the trace of the Frobenius at 2, and obtain Table 2. Note that the traces of Frobenius for ρ_4 are the same as those for ρ_2, with the values of ζ and ζ^2 swapped.

Since 83 is unramified in L/K, it has ramification index 2 in $L/\mathbb{Q}$. It is then easy to see (using the formula for the level) that the level of ρ_2 and ρ_4 is 83 (and the nebentype ϵ is the unique quadratic character modulo 5 ramified only at 83), and that the level of ρ_1 and ρ_3 is 83^2 (with trivial nebentype). Since level 83^2 is too large for our programs to deal with, we will concentrate on ρ_2 and ρ_4.

The ramification index of 5 in $L/\mathbb{Q}$ is 3, and the inertia group is easily seen to be noncentral. Since 3 does not divide $5-1=4$, but does divide $5^2-1=24$, all of the representations ρ_i are niveau 2. Restricting to inertia at 5, we see that

$$\rho_i|_{I_5} \sim \begin{pmatrix} \psi^8 & & \\ & {\psi'}^8 & \\ & & \omega^0 \end{pmatrix},$$

where $8=3+1*5$. Thus, a triple (a_1,a_2,a_3) associated to this representation is $(3,1,0)$, and a predicted weight is $F(3-2,1-1,0)'$. This weight is not uniquely defined–the conjecture predicts that a cohomology eigenclass with ρ_i attached should exist in at least one of the weights $F(1,0,0)$ or $F(5,4,0)$.

Other weights are also predicted. For instance, if we permute the diagonal characters (by conjugating ρ), we see that

$$\rho|_{I_5} \sim \begin{pmatrix} \psi^8 & & \\ & \omega^0 & \\ & & {\psi'}^8 \end{pmatrix},$$

yielding a triple of $(3,0,1)$ and a predicted weight of

$$F(3-2,0-1,1)' = F(5,3,1) = F(4,2,0) \otimes \det{}^1.$$

We may also permute the order of the two characters of niveau 2. Recalling that $\psi' = \psi^5$, we see that ${\psi'}^8 = \psi^{16}$, so we have that

$$\rho_i|_{I_5} \sim \begin{pmatrix} \psi^{16} & & \\ & {\psi'}^{16} & \\ & & \omega^0 \end{pmatrix},$$

with $16 = 6+2*5$, and the predicted weight is $F(6-2,2-1,0) = F(4,1,0)$.

Other permutations of the diagonal characters give predicted weights of $F(4,3,0)\otimes\det^2$, $F(2,1,0)\otimes\det^2$, and at least one of $F(1,1,0)\otimes\det^1$ or $F(5,1,0)\otimes\det^1$. (The last two come from the same triple, as in the weight $F(1,0,0)$ case.)

Using the techniques described in the next section, we have shown that for $V = F(1,0,0), F(1,1,0)\otimes\det, F(4,2,0)\otimes\det, F(4,1,0), F(4,3,0)\otimes\det^2$, and $F(2,1,0)\otimes\det^2$, the cohomology $H^3(\Gamma_0(83), V(\epsilon))$ contains cohomology eigenclasses which have the correct eigenvalues (at least for primes up to 47) to have ρ_2 and ρ_4 attached. In fact, each cohomology group contains two one-dimensional eigenspaces, one with the correct eigenvalues to have ρ_2 attached, and one with the correct eigenvalues to have ρ_4 attached. These eigenspaces are defined over $\mathbb{F}_{25}$, and conjugate over $\mathbb{F}_5$, as we would expect. Hence, our calculations give evidence that the weights predicted by Conjecture 2.5 are correct.

4 Computational Techniques

In our computations to verify a specific case of the conjecture, we make use of the natural duality between cohomology and homology—in fact we always compute homology groups.

The basic idea behind the computations is the same as in [1]. A variant of Theorem 2.1 of [1] allows us to find the homology of $SL_3(\mathbb{Z})$ with coefficients in any given $\mathrm{GL}_3(\mathbb{F}_p)$-module. However, we are interested in finding the homology of congruence subgroups $\Gamma_0(N)$. Following [3], we use the Hecke equivariance (see [5]) of the Shapiro isomorphism

$$H_n(\Gamma_0(N), V(\epsilon)) \cong H_n(\mathrm{SL}_3(\mathbb{Z}), \mathrm{Ind}_{\Gamma_0(N)}^{\mathrm{SL}_3(\mathbb{Z})} V(\epsilon))$$

to compute these homology groups. Hence, all of our computations are reduced to calculating homology of $\mathrm{SL}_3(\mathbb{Z})$ with coefficients in various modules.

Once we have the homology groups calculated, we use exactly the same method as in [1] to calculate the Hecke eigenvalues. In fact, rather than recompute the unimodular matrices described there, we have used the same files generated in the course of [1].

Note that we have improved the efficiency of the algorithms described in [1] by converting their programs from *Mathematica* into C, and by computing the relevant matrices one row at a time.

One further innovation which we have implemented involves the use of a filtration of standard $\mathrm{GL}_3(\mathbb{F}_p)$-modules to isolate homology eigenclasses coming from specific irreducible modules. This is an improvement over the computations done in [4] and [1], where the computations were done only over the modules V_g, and conclusions about irreducible subquotients of V_g were difficult to make. We begin by defining V_g, together with a certain filtration of submodules.

Definition 4.1. V_g is the $\mathrm{GL}_3(\mathbb{F}_p)$-module of homogeneous polynomials of degree 3 in three variables over F_p, with the standard action of $\mathrm{GL}_3(\mathbb{F}_p)$. Taking the three variables to be x, y, and z, we set

$$V_{g,i} = \mathrm{Span}\{x^a y^b z^c \in V_g : \lfloor a/p \rfloor + \lfloor b/p \rfloor + \lfloor c/p \rfloor \geq i\}.$$

We note that by "freshman exponentiation", $V_{g,i}$ is a submodule of V_g. Taking the quotients $W_{g,i} = V_{g,i}/V_{g,i+1}$ gives us modules which, although not irreducible, contain many fewer irreducible subquotients than the V_g themselves. In many cases, we can find all the composition factors of the $W_{g,i}$ and use this information to show that certain eigenvalues must come from specific irreducible modules. The techniques to do this are somewhat ad hoc, and we illustrate by means of specific examples.

To begin with, we note that Table 4 in [7] allows us to determine all the composition factors of V_g, including multiplicity (although it does not allow us to determine whether a given component is a submodule or a subquotient). We are then interested in using this decomposition to determine the composition factors of the $W_{g,i}$. Now the dimension of V_g is $(g+1)(g+2)/2$, and the module $F(a,b,0)$ is a subquotient of $V_{(a-b)} \otimes V_b^*$ (where the * denotes the dual). Hence, we know that the dimension of $F(a,b,0)$ is at most $(a-b+1)(a-b+2)(b+1)(b+2)/4$. For example, we find that we have the decompositions of V_5, V_6, and V_7 (modulo 5) given in Table 3.

Explicit computation with $g = 5, 6, 7$ and $p = 5$, show that $V_{5,1}$ has dimension 3, $V_{6,1}$ has dimension 9, and $V_{7,1}$ has dimension 18. The above

Module	Dimension	Module	Dimension	Module	Dimension
V_5	21	V_6	28	V_7	36
$F(1,0,0)$	3	$F(2,0,0)$	6	$F(3,0,0)$	10
		$F(1,1,0)$	≤ 3	$F(2,1,0)$	≤ 9
$F(4,1,0)$	18	$F(4,2,0)$	≤ 36	$F(4,3,0)$	≤ 30

Table 3. Decomposition of certain modules

decompositions of V_5 and V_6 then allow us to deduce that $V_{5,1} \cong F(1,0,0)$ and $V_{6,1}$ has composition factors $F(2,0,0)$ and $F(1,1,0)$. We also find that the dimension of $F(1,1,0)$ is three, and the dimension of $F(4,2,0)$ is 19. Then we immediately deduce that $W_{5,0} \cong F(4,1,0)$ and $W_{6,0} \cong F(4,2,0)$. We also find immediately that one of $V_{7,1}$ and $W_{7,0}$ is isomorphic to $F(4,3,0)$ and the other has composition factors $F(2,1,0)$ and $F(3,0,0)$ (as well as the fact that $F(2,1,0)$ has dimension 8 and $F(4,3,0)$ has dimension 18), although determining which structure is associated with which module is more difficult (it turns out that $W_{7,0}$ is isomorphic to $F(4,3,0)$, although this fact is not needed for our purposes).

We may now easily compute directly with the modules $F(1,0,0)$, $F(4,2,0)$, and $F(4,1,0)$. In addition, if a certain system of Hecke eigenvalues occurs in homology with coefficients in $V_{6,1}$, but not in homology with weight $F(2,0,0)$, then we know that this system of eigenvalues must arise in weight $F(1,1,0)$ (since by [5] every system of eigenvalues occurring for some module must come from some irreducible subquotient).

Similarly, a system of eigenvalues occurring in both $V_{7,1}$ and $W_{7,0}$, but not in $F(3,0,0) \cong V_3$ must occur in both $F(2,1,0)$ and $F(4,3,0)$.

For our example, we computed the homology of $\Gamma_0(83)$, with nebentype ϵ, and coefficient modules $F(1,0,0)$, $W_{5,0} \cong F(4,1,0)$, $W_{6,0} \otimes \det \cong F(4,2,0) \otimes \det$, $V_{6,1} \otimes \det$, $V_{7,1} \otimes \det^2$, and $W_{7,0} \otimes \det^2$, and found that the correct eigenvalues existed in all of these cohomology groups. We also computed the cohomology in weights $F(2,0,0) \otimes \det$ and $F(3,0,0) \otimes \det^2$, and did not find the correct eigenvalues. Hence, we have confirmed that in the weights predicted by Conjecture 2.5, eigenclasses exist with the correct eigenvalues (up to $\ell = 47$) to have ρ_2 and ρ_4 attached.

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References

[1] G. Allison, A. Ash, and E. Conrad, *Galois representations, Hecke operators, and the mod-p cohomology of* $\mathrm{GL}(3,\mathbb{Z})$ *with twisted coefficients*, Experiment. Math. **7** (1998), 361–390.

[2] A. Ash, *Galois representations attached to mod p cohomology of* $GL(n,\mathbb{Z})$, Duke Math. J. **65** (1992), 235–255.

[3] A. Ash, D. Doud, and D. Pollack, *Galois representations with conjectural connections to arithmetic cohomology*, Duke Math. J., to appear.

[4] A. Ash and W. Sinnott, *An analogue of Serre's conjecture for Galois representations and Hecke eigenclasses in the mod-p cohomology of* $GL(n,\mathbb{Z})$, Duke Math. J. **105** (2000), 1–24.

[5] A. Ash and G. Stevens, *Cohomology of arithmetic groups and congruences between systems of hecke eigenvalues*, J. Reine Angew. Math. (1986), 192–220.

[6] A. Ash and P. Tiep, *Modular representations of* $GL(3,\mathbb{F}_p)$*, symmetric squares, and mod-p cohomology of* $GL(3,\mathbb{Z})$, J. Algebra **222** (1999), 376–399.

[7] S. R. Doty and G. Walker, *The composition factors of* $\mathbb{F}_p[x_1,x_2,x_3]$ *as a* $\mathrm{GL}(3,p)$*-module*, J. Algebra **147** (1992), 411–441.

[8] J.-P. Serre, *Sur les représentations modulaires de degré* 2 *de* $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, Duke Math. J. **54** (1987), 179–230.

[9] B. F. Wyman, *Hilbert class fields and group extensions*, Scripta Math. **29** (1973), 141–149.