Handout 1
Population Sequences

This handout deals with sequences produced by a simple mathematical model from population biology.

The Model

Let P_n be the population of a species in year n. A simple model for how P_{n+1} might depend on P_k for $k \leq n$ is given by the recurrence relation

$$P_{n+1} = P_n + (b - d)P_n. \tag{1}$$

Here b and d are constants that represent, respectively, the natural birth and death rates of the species. We assume, here and below, that $b > d$. It is not hard to see that this assumption implies that a population obeying the growth law (1) grows without bound (if the initial population P_1 is positive). Furthermore, for even modest values of $b - d$, the population becomes incredibly large after a short number of years.

A more realistic model might take into account the fact that environmental conditions typically impose a limit on sustainable population size. If we let K be the *carrying capacity* of the environment, one way to modify (1) to reflect these limitations is:

$$P_{n+1} = P_n + \frac{K - P_n}{K}(b - d)P_n. \tag{2}$$

Note that the new factor $(K - P_n)/K$ in (2):

- converges to 1 as P_n goes to 0,
- is 0 when $P_n = K$, and
- is negative when $P_n > K$.

These properties reflect the fact that environmental limitations have negligible effect when the population is small, that these limitations halt population growth completely when the carrying capacity is reached, and that if the carrying capacity should ever be exceeded the population will decrease in the next year. From now on, we assume that $\{P_n\}$ satisfies (2).

Exercise 1

(a) There is a constant E larger than K that has the property that $P_{n+1} = 0$ whenever $P_n = E$. (E is called the *extinction level*, because if the population should ever reach E, the species will be extinct the next year.) Determine E in terms of K, b, and d.

(b) For each natural number n, let $x_n = P_n/E$. (We call x_n the *normalized* population.) Show that the sequence $\{x_n\}$ satisfies the recurrence relation

$$x_{n+1} = \mu x_n(1 - x_n) \tag{3}$$

for some constant μ, and determine the value of this constant in terms of K, b, and d.

From now on, we focus on (3) and assume that $\{x_n\}$ satisfies that recurrence relation. If we define

$$f_\mu(x) := \mu x(1-x),$$

then (3) becomes $x_{n+1} = f_\mu(x_n)$.

Convergent Populations

Theorem. If $\lim_{n\to\infty} x_n = L$, then $f_\mu(L) = L$.
Proof. If $\lim_{n\to\infty} x_n = L$, then (since f_μ is continuous)

$$\lim_{n\to\infty} f_\mu(x_n) = f_\mu(L). \tag{4}$$

But $f_\mu(x_n) = x_{n+1}$, so

$$\lim_{n\to\infty} f_\mu(x_n) = \lim_{n\to\infty} x_{n+1} = L \tag{5}$$

(since the limit of a sequence depends only on the tail of the sequence). Combining (4) and (5) gives $f_\mu(L) = L$. $\square$

It is easy to check that the only solutions to $f_\mu(L) = L$ are $L = 0$ and $L = 1 - 1/\mu$. Thus, if the unnormalized population P_n approaches a constant as $n \to \infty$, then that constant must be either 0 or $(1 - 1/\mu)E$.

Definition. A sequence $\{x_n\}$ is *eventually nondecreasing* if there is a natural number k such that $x_{n+1} \geq x_n$ whenever $n \geq k$.

It is a fact that a sequence that is bounded above and eventually nondecreasing converges (to the least upper bound of $\{x_k, x_{k+1}, x_{k+2}, \ldots\}$).

Exercise 2 Suppose that $1 < \mu \leq 2$.
(a) Show that if $x_n \in (0, 1 - 1/\mu]$ for some n, then (for that same n) $x_{n+1} \in (0, 1 - 1/\mu]$ and $x_{n+1} \geq x_n$.
(b) Show that if $x_1 \in (0, 1 - 1/\mu]$ then $\lim_{n\to\infty} x_n = 1 - 1/\mu$. (Hint: Show that $\{x_n\}$ is bounded above and nondecreasing.)
(c) Show that if $x_1 \in [1 - 1/\mu, 1)$ then $\lim_{n\to\infty} x_n = 1 - 1/\mu$. (Hint: Show that $\{x_n\}$ is either nonincreasing or is eventually nondecreasing.)

Periodic Populations

Combining parts (b) and (c) of Exercise 2, we see that for $\mu \in (1, 2]$ if the population P_n in the first year is anywhere between 0 and E then the population approaches a constant as time passes (and the limiting population doesn't depend on the specific value of the starting population).

If $\mu > 2$, on the other hand, it may be the case that the population doesn't eventually approach a constant.

Exercise 3 Suppose that $\{x_n\}$ satisfies the recurrence relation (3). Show that if $\mu \in [1, 4)$ and $x_1 \in (0, 1)$ then $x_n \in (0, 1)$ for every natural number n.

Exercise 3 shows that if $\mu \in [1, 4)$ and the starting population is below the extinction level E then the population remains below the extinction level (and positive) in all subsequent years (even if the population doesn't approach a constant).

Definition. We say that a sequence $\{x_n\}$ is *k-periodic* if $x_{n+k} = x_n$ for every natural number n.

It is a fact that a sequence generated by a recurrence relation like (3) is k-periodic if $x_{1+k} = x_1$.

If $\mu > 3$, then $\{x_n\}$ may be k-periodic without being constant. For example, suppose $\mu = 7/2$ and $x_1 = 3/7$. Then

$$x_2 = f_{7/2}(3/7) = \frac{7}{2} \cdot \frac{3}{7}\left(1 - \frac{3}{7}\right) = \frac{3}{2} \cdot \frac{4}{7} = \frac{6}{7},$$

and

$$x_3 = f_{7/2}(6/7) = \frac{7}{2} \cdot \frac{6}{7}\left(1 - \frac{6}{7}\right) = 3\left(\frac{1}{7}\right) = \frac{3}{7} = x_1,$$

so $\{x_n\}$ is 2-periodic.

Chaotic Populations

If $\mu \in (3.57, 4)$ and x_1 is chosen at random in $(0, 1)$, then the resulting population sequence $\{x_n\}$ is likely to be *chaotic*. This means that it manifests no discernible pattern: It doesn't diverge to ∞, it doesn't diverge to $-\infty$, it doesn't converge to a constant, it isn't k-periodic, and it doesn't get closer and closer to a k-periodic sequence.

For example, if $\mu = 3.7$ and $x_1 = 0.3$, then the plot of the population sequence $\{x_n\}$ looks like: