STABILITY OF ISENTROPIC NAVIER–STOKES SHOCKS

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ABSTRACT. We announce recent results obtained through a combination of asymptotic ODE estimates and numerical Evans function calculations, which together yield stability of isentropic Navier–Stokes shocks for a γ-law gas with $1 \leq \gamma \leq [1, 2.5]$. Other γ may be treated similarly.

1. INTRODUCTION

The isentropic compressible Navier-Stokes equations in one spatial dimension expressed in Lagrangian coordinates take the form

$$\begin{aligned} &v_t - u_x = 0, \\ &u_t + p(v)_x = \left(\frac{u_x}{v}\right)_x, \end{aligned} \tag{1}$$

where v is specific volume, u is velocity, and p pressure. We assume an adiabatic pressure law $p(v) = a_0 v^{-\gamma}$ corresponding to a γ-law gas, for some constants $a_0 > 0$ and $\gamma \geq 1$.

These equations are well known to support "viscous shock layers", or asymptotically-constant traveling-wave solutions

$$(v, u)(x, t) = (\hat{v}, \hat{u})(x - st), \qquad \lim_{z \to \pm\infty} (\hat{v}, \hat{u})(z) = (v, u)_{\pm}. \tag{2}$$

In nature, such waves are seen to be quite stable, even for large variations in pressure between $v_{\pm}$. It is a fundamental question whether and to what extent this is reflected in the continuum-mechanical model (1), that is, for which choice of parameters $(v, u)_{\pm}$, γ solutions (2) are stable in the sense of time-evolutionary PDE.

Substantial progress in the form of "Lyapunov-type" theorems established in [5, 8] has reduced the problem of linearized and nonlinear stability to determination of spectral stability, i.e., the study of the associated eigenvalue ODE. However, until recently, the only results on the spectral stability problem were for small-amplitude shocks [6, 4] or the special case $\gamma = 1$ [6, 5], with the large-amplitude case remaining essentially open.

The purpose of this note is to announce the resolution of this problem in [1, 3] by a combination of asymptotic ODE and numerical Evans function computations: specifically, the result of unconditional stability of arbitrary-amplitude isentropic Navier–Stokes shocks for $1 \leq \gamma \leq 2.5$. Other γ may be treated by the same methods but were not considered.

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2. The rescaled equations

Taking $(x, t, v, u, a_0) \to (-\varepsilon s(x - st), \varepsilon s^2 t, v/\varepsilon, -u/(\varepsilon s), a_0 \varepsilon^{-\gamma-1} s^{-2})$, with ε so that $0 < v_+ < v_- = 1$, we consider stationary solutions $(\hat{v}, \hat{u})(x)$ of

$$
\begin{aligned}
&v_t + v_x - u_x = 0,\\
&u_t + u_x + (av^{-\gamma})_x = \left(\frac{u_x}{v}\right)_x.
\end{aligned} \tag{3}
$$

2.1. **Profile equation.** Steady shock profiles of (3) satisfy

$$
v' = H(v, v_+) := v(v - 1 + a(v^{-\gamma} - 1)), \tag{4}
$$

where a is found by $H(v_+, v_+) = 0$, yielding the Rankine-Hugoniot condition

$$
a = -\frac{v_+ - 1}{v_+^{-\gamma} - 1} = v_+^\gamma \frac{1 - v_+}{1 - v_+^\gamma}. \tag{5}
$$

Evidently, $a \to \gamma^{-1}$ in the weak shock limit $v_+ \to 1$, while $a \sim v_+^\gamma$ in the strong shock limit $v_+ \to 0$. In this scaling, the large-amplitude limit corresponds to the limit as $v_+ \to 0$, or density $\rho_+ := 1/v_+ \to \infty$.

2.2. **Eigenvalue equations.** Linearizing (3) about the profile $(\hat{v}, \hat{u})$ and integrating with respect to x, we obtain the integrated eigenvalue problem

$$
\lambda v + v' - u' = 0, \tag{6a}
$$

$$
\lambda u + u' - \frac{h(\hat{v})}{\hat{v}^{\gamma+1}} v' = \frac{u''}{\hat{v}}, \tag{6b}
$$

where $h(\hat{v}) = -\hat{v}^{\gamma+1} + a(\gamma - 1) + (a + 1)\hat{v}^\gamma$. Spectral stability of $\hat{v}$ corresponds to nonexistence of solutions of (6) decaying at $x = \pm\infty$ for $\Re e\lambda \geq 0$ [4, 1, 3].

3. Preliminary estimates

Proposition 3.1 ([1]). *For each $\gamma \geq 1$, $0 < v_+ \leq 1$, (4) has a unique (up to translation) monotone decreasing solution $\hat{v}$ decaying to its endstates with a uniform exponential rate. For $0 < v_+ \leq \frac{1}{12}$ and $\hat{v}(0) := v_+ + \frac{1}{12}$,*

$$
|\hat{v}(x) - v_+| \leq \left(\frac{1}{12}\right) e^{-\frac{3x}{4}} \quad x \geq 0, \tag{7a}
$$

$$
|\hat{v}(x) - v_-| \leq \left(\frac{1}{4}\right) e^{\frac{x+12}{2}} \quad x \leq 0. \tag{7b}
$$

Proof. Existence and monotonicity follow trivially by the fact that (4) is a scalar first-order ODE with convex righthand side. Exponential convergence as $x \to +\infty$ follows by $H(v, v_+) = (v - v_+)\Big(v - \left(\frac{1-v_+}{1-v_+^\gamma}\right)\Big(\frac{1-\left(\frac{v_+}{v}\right)^\gamma}{1-\left(\frac{v_+}{v}\right)}\Big)\Big)$, whence $v - \gamma \leq \frac{H(v,v_+)}{v-v_+} \leq v - (1 - v_+)$ by $1 \leq \frac{1-x^\gamma}{1-x} \leq \gamma$ for $0 \leq x \leq 1$. See [1]. □

Proposition 3.2 ([6]). *Viscous shocks of* (1) *are spectrally stable whenever* $\big(\frac{v_+^{\gamma+1}}{a\gamma}\big)^2 + 2(\gamma - 1)\big(\frac{v_+^{\gamma+1}}{a\gamma}\big) - (\gamma - 1) \geq 0$, *in particular, for* $|v_+ - 1| << 1$.

Proof. Writing (6) as $U_t + A(x)U_x = B(x)U_{xx}$, with $A = \begin{pmatrix} 1 & -1 \\ -\frac{h(\hat v)}{\hat v^{\gamma+1}} & 1 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{\hat v} \end{pmatrix}$, we see that $S = \begin{pmatrix} 1 & 0 \\ 0 & \frac{\hat v^{\gamma+1}}{h(\hat v)} \end{pmatrix}$ symmetrizes A, B. Taking the L^2 complex inner product of SU against the equations yields

$$\Re e\lambda \langle U, SU\rangle + \langle U', SBU'\rangle = -\langle u, g(\hat v)u\rangle,$$

where the righthand side, coming from commutator terms, is of favorable sign for v_+ satisfying the condition of the proposition. See [6, 1]. □

Proposition 3.3 ([1])**.** *Nonstable eigenvalues λ of (6), i.e., eigenvalues with nonnegative real part, are confined for any $\gamma \geq 1$, $0 < v_+ \leq 1$ to the region Λ defined by*

$$\Re e(\lambda) + |\Im m(\lambda)| \leq \Big(\sqrt{\gamma} + \frac{1}{2}\Big)^2. \tag{8}$$

Proof. Energy estimates related to those of Proposition 3.2. See [1]. □

4. Evans function formulation

Following [1], we may express (6) as a first-order system $W' = A(x,\lambda)W$,

$$A(x,\lambda) = \begin{pmatrix} 0 & \lambda & 1 \\ 0 & 0 & 1 \\ \lambda\hat v & \lambda \hat v & f(\hat v) - \lambda \end{pmatrix}, \quad W = \begin{pmatrix} u \\ v \\ v' \end{pmatrix}, \quad ' = \frac{d}{dx}, \tag{9}$$

$$f(\hat v) = 2\hat v - (\gamma - 1)\Big(\frac{1 - v_+}{1 - v_+^\gamma}\Big)\Big(\frac{v_+}{\hat v}\Big)^\gamma - \Big(\frac{1 - v_+}{1 - v_+^\gamma}\Big)v_+^\gamma - 1. \tag{10}$$

Eigenvalues of (6) correspond to nontrivial solutions W for which the boundary conditions $W(\pm\infty) = 0$ are satisfied. Because $A(x,\lambda)$ as a function of $\hat v$ is asymptotically constant in x, the behavior near $x = \pm\infty$ of solutions of (9) is governed by the limiting constant-coefficient systems

$$W' = A_\pm(\lambda)W, \qquad A_\pm(\lambda) := A(\pm\infty, \lambda). \tag{11}$$

We readily find on the (nonstable) domain $\Re e\lambda \geq 0$, $\lambda \neq 0$ of interest that there is a one-dimensional unstable manifold $W_1^-(x)$ of solutions decaying at $x = -\infty$ and a two-dimensional stable manifold $W_2^+(x) \wedge W_3^+(x)$ of solutions decaying at $x = +\infty$, analytic in λ, with asymptotic behavior

$$W_j^\pm(x,\lambda) \sim e^{\mu_\pm(\lambda)x} V_j^\pm(\lambda) \tag{12}$$

as $x \to \pm\infty$, where $\mu_\pm(\lambda)$ and $V_j^\pm(\lambda)$ are eigenvalues and associated analytically chosen eigenvectors of the limiting coefficient matrices $A_\pm(\lambda)$.

A standard choice of eigenvectors $V_j^\pm$ [2], uniquely specifying D (up to constant factor) is obtained by Kato's ODE, a linear, analytic ODE whose solution can be alternatively characterized by the property that there exist corresponding left eigenvectors $\tilde V_j^\pm$ such that, denoting $d/d\lambda$ by "$\cdot$"

$$(\tilde V \cdot V)^\pm \equiv \text{constant}, \quad (\tilde V \cdot \dot V)^\pm \equiv 0, \tag{13}$$

Defining the *Evans function* D associated with operator L as

$$D(\lambda) := \det(W_1^- W_2^+ W_3^+)_{|x=0}, \tag{14}$$

we find that D is analytic on $\Re e\lambda \geq 0$, with eigenvalues of L corresponding in location and multiplicity to zeroes of D. See [8] for further details.

4.1. **Example.** Consider Burgers' equation, $u_t + (u^2)_x = u_{xx}$, and the family of stationary viscous shock solutions

$$\hat{u}^\epsilon(x) := -\epsilon \tanh(\epsilon x/2), \qquad \lim_{z\to\pm\infty} \hat{u}^\epsilon(z) = \mp\epsilon, \tag{15}$$

with associated integrated eigenvalue equations $w'' = \hat{u}^\epsilon w' + \lambda w$. By a linearized Hopf–Cole transformation [3], we may compute the associated Evans functions explicitly to be not only stable but *identically constant*,

$$D_\epsilon(\lambda) \equiv -2\sqrt{\epsilon^2/4 + 1}, \tag{16}$$

and converging in the weak shock limit $\epsilon \to 0$ to a nonzero constant. Burgers' equation models behavior in the weak shock limit of general systems; see, e.g., [7].

5. Main results

5.1. **The strong shock limit.** Taking a formal limit as $v_+ \to 0$ of the rescaled equations (3) and recalling that $a \sim v_+^\gamma$, we obtain a limiting evolution equation

$$\begin{aligned} v_t + v_x - u_x &= 0,\\ u_t + u_x &= \Big(\frac{u_x}{v}\Big)_x \end{aligned} \tag{17}$$

corresponding to a *pressure-less gas*, or $\gamma = 0$.

The associated limiting profile equation $v' = v(v-1)$ has explicit solution $\hat{v}_0(x) = \frac{1-\tanh(x/2)}{2}$; the limiting eigenvalue system is $W' = A^0(x,\lambda)W$,

$$A^0(x,\lambda) = \begin{pmatrix} 0 & \lambda & 1\\ 0 & 0 & 1\\ \lambda\hat{v}_0 & \lambda\hat{v}_0 & f_0(\hat{v}_0) - \lambda \end{pmatrix}, \tag{18}$$

where $f_0(\hat{v}_0) = 2\hat{v}_0 - 1 = -\tanh(x/2)$.

Observe that the limiting coefficient matrix $A^0_+(\lambda) := A^0(+\infty,\lambda)$ is nonhyperbolic (in ODE sense) for all λ, having eigenvalues $0, 0, -1-\lambda$; in particular, the stable manifold drops to dimension one in the limit $v_+ \to 0$, and so the prescription of an associated Evans function is *underdetermined*.

This difficulty is resolved by a careful boundary-layer analysis in [3], determining a special "slow stable" mode $V_2^+ \pm (1,0,0)^T$ augmenting the "fast stable" mode $V_3 := (a^{-1}(\lambda/a+1), a^{-1}, 1)^T$ associated with the single stable eigenvalue $a = -1-\lambda$ of A^0_+. This determines a *limiting Evans function* $D^0(\lambda)$ by the prescription (14), (12) of Section 4.

Theorem 5.1 ([3]). *For λ in any compact subset of $\Re e\lambda \geq 0$, $D(\lambda)$ converges uniformly to $D^0(\lambda)$ as $v_+ \to 0$.*

Proof. Careful boundary layer analysis/asymptotic ODE estimates [3, 7]. □

Proposition 5.2 ([3]). *The limiting function D^0 is nonzero on $\Re e\lambda \geq 0$.*

Proof. Energy estimate adapted from that of Proposition 3.2. □

Corollary 5.3. *For any $\gamma \geq 1$, isentropic Navier–Stokes shocks are stable in the strong shock limit, i.e., for v_+ sufficiently small.*

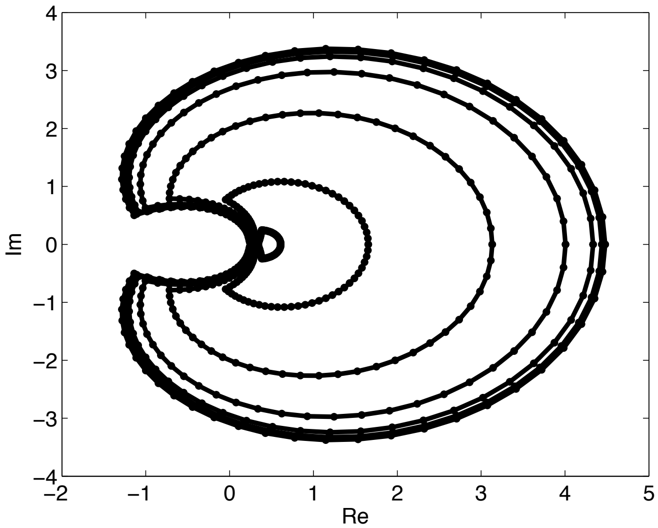

FIGURE 1. Convergence to the limiting Evans function as $v_+ \to 0$ for a monatomic gas, $\gamma = 5/3$. The contours depicted, going from inner to outer, are images of the semicircle under D for $v_+ =$`1e-1,1e-2,1e-3,1e-4,1e-5,1e-6`. The outermost contour is the image under D^0, which is nearly indistinguishable from the image for $v_+ =$`1e-6`.

5.2. **The weak shock limit.** Stability in the weak shock limit is known [6]. However, combining the calculation of Section 4.1 with asymptotic ODE estimates estimates of [7], we obtain the new observation that the Evans function converges in the weak shock limit to a *constant function*.

5.3. **Intermediate strength shocks.** Having disposed analytically of the weak and strong shock limits, we have reduced the problem of shock stability to a bounded parameter range on which the Evans function may be effiently computed numerically, in uniformly well-conditioned fashion; see [2]. Specifically, we may map a semicircle $\partial\{\Re e\lambda \geq 0\} \cap \{|\lambda| \leq 10\}$ enclosing Λ for $\gamma \in [1, 3]$ by D^0 and compute the winding number of its image about the origin to determine the number of zeroes of D^0 within the semicircle, and thus within Λ. For details of the numerical algorithm, see [1, 2].

Such a study was carried out systematically in [1] on the parameter range $\gamma \in [1, 3]$, for shocks with Mach number $M \in [1, 3{,}000]$, which corresponds on $\gamma \in [1, 2.5]$ to $v_+ \geq 10^{-3}$, with the result of stability for all values tested. In combination with the results of Sections 5.1 and 5.2, this gives convincing numerical evidence, as claimed, of *unconditional stability of isentropic Navier–Stokes shocks for* $\gamma \in [1, 2.5]$ *and arbitrary amplitude.*

5.4. **Global picture.** In Figure 1, we superimpose on the numerically computed image of the semicircle by D^0 its (numerically computed) image by the full Evans function D, for a monatomic gas $\gamma \approx 1.66$ at successively higher Mach numbers

$v+ =$1e-1,1e-2,1e-3,1e-4,1e-5,1e-6, showing both convergence of D to D^0 in the strong shock limit as v_+ approaches zero and convergence of D to a constant in the weak shock limit $v_+ \to 1$.

Moreover, the displayed contours are, to the scale visible by eye, "monotone" in v_+, or nested, one within the other, with lower-Mach number contours are essentially "trapped" within higher-Mach number contours, and all contours interpolating smoothly between this and the inner, constant limit. Behavior for other $\gamma \in [1, 3]$ is entirely similar; see [3]. That is, a great deal of topological information is encoded in the analytic family of Evans functions indexed by v_+, from which stability may be deduced almost by inspection. Such topological information does not seem to be available from other methods of investigating stability such as direct discretation of the linearized operator about the wave, studies based on linearized time-evolution, or power methods, and represents in our view a significant advantage of the Evans function formulation.

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