NORMALIZATION OF RESONANT HAMILTONIANS

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INTRODUCTION

The definition of integrability is simple to state; an autonomous N degree of freedom Hamiltonian is integrable if N independent global invariants exist and these are in involution with each other.[1] However, a failure to find such a set of global invariants does not exclude the possibility that the Hamiltonian in question is integrable. The detection of integrability is thus a critical issue in non-linear dynamics and a variety of analytical and numerical procedures has been developed to determine if a Hamiltonian is integrable. The most obvious approach is to try to establish if the Hamiltonian is separable, possibly using the Stäckel conditions to guide one to appropriate coordinate system. It should be noted, however, that finding coordinates that separate a particular problem can itself be a difficult task. More general and systematic approaches than simply seeking separability are therefore in order, e.g., the Whittaker program. An alternative method is the Painlevé test[2] in which the analytic structure of the equations of motion in the complex time plane is examined. This approach has been used to uncover integrability but must be applied gingerly because it cannot be guaranteed to succeed in every case. Probably the simplest numerical method is to generate Poincaré surfaces of section and determine by eye whether or not the motion is integrable. Of course, no numerical method by itself can definitively determine integrability.

Deprit and co-workers[3–7] in their studies of normal forms, and particularly in an application to the Toda Lattice[3] have discovered what might constitute a new symptom of integrability in Hamiltonian systems. They note a correlation between the persistence of degeneracy in the normal form to high order and integrability of the pre-normalized Hamiltonian, and conjecture that this might be a symptom of integrability.[6] However, tests of this conjecture have ultimately uncovered integrable systems that are also separable. In a recent study of a problem in atomic physics we have encountered a Hamiltonian possessing a non-separable, integrable limit.[7] Surprisingly, in this case the conjecture of Deprit and Miller seems not to hold which

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led us to question whether normalization might sense separability rather than integrability *per se.* After all, it is well known quantum mechanically that separability leads to degeneracies and degenerate equilibria are the symptom that the normal form is supposed to exhibit in the case of integrability. In this paper we examine normalization of, *(i)* a class of perturbed isotropic oscillators to high order which admit various integrable limits, and, *(ii)* the hydrogen atom in a generalized van der Waals (GVDW) potential. We conclude that under certain circumstances normalization can detect separability, but that it may also overlook integrable cases, whether separable or not.

NORMALIZATION OF ELLIPTIC OSCILLATORS

The normalization of perturbed elliptic oscillators has been studied thoroughly by Deprit and co-workers.[3–7] In fact, our exhibition problem is the same as Miller's,[6] i.e., a perturbed elliptic oscillator of the form $\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_1$ where

$$\mathcal{H}_0 = \frac{1}{2}(P_x^2 + P_y^2) + \frac{1}{2}(x^2 + y^2) \tag{1}$$

and

$$\mathcal{H}_1 = \epsilon(\alpha x^3 + \beta x y^2) \tag{2}$$

High order normalizations of (i.e., 1:1 resonant) elliptic oscillators perturbed by quartic and sextic polynomials have also been investigated recently: for these systems the integrable limits also turn out to be globally separable.[4–8] To start it is useful to consider the properties of the isotropic oscillator $\mathcal{H}_0$, in particular its integrals of motion. Including the energy there are four invariants

$$\pi_0 = E_0/2 = \frac{1}{4}(P_x^2 + x^2) + \frac{1}{4}(P_y^2 + y^2)$$

$$\pi_1 = D/2 = \frac{1}{4}(P_x^2 + x^2) - \frac{1}{4}(P_y^2 + y^2)$$

$$\pi_2 = L/2 = \frac{1}{2}(yP_x - xP_y)$$

$$\pi_3 = K/2 = \frac{1}{2}(P_x P_y + xy). \tag{3}$$

The $\{\pi_i, i = 1,2,3\}$ are called the *Hopf* variables and are related in a simple fashion to D, L, K which correspond to the more usual definitions of the invariants of the isotropic oscillator. The Hopf variables have the useful property of satisfying the same Poisson bracket relations as angular momentum, namely,

$$\{\pi_j, \pi_k\} = \epsilon_{jkl}\pi_l \tag{4}$$

where ϵ_{jkl} is equal to 1 (-1) for even (odd) permutations of its subscripts and to 0 otherwise. Together with the relation

$$\pi_0^2 = \pi_1^2 + \pi_2^2 + \pi_3^2 \tag{5}$$

the $\{\pi_i, i = 1,2,3\}$ generate the Lie algebra of the group $SU(2)$. Equation (5) is that of a sphere, sometimes nominated the Poincaré sphere, and on whose surface the phase flow of the reduced system can be portrayed.

Further geometrical insight can be obtained by recognizing that the Hopf variables may be transformed to action-angle variables j_x, j_y, ϕ_x, ϕ_y as follows,

$$x = \sqrt{2j_x}\sin\phi_x, \quad y = \sqrt{2j_y}\sin\phi_y$$

$$P_x = \sqrt{2j_x}\cos\phi_x, \quad P_y = \sqrt{2j_y}\cos\phi_y \tag{6}$$

After a further transformation

$$J_1 = \frac{(j_x + j_y)}{2}, \quad J_2 = \frac{(j_x - j_y)}{2}$$

$$\phi_1 = \phi_x + \phi_y, \quad \phi_2 = \phi_x - \phi_y \tag{7}$$

the $\{\pi_i\}$ become,

$$\pi_0 = J_1$$

$$\pi_1 = J_2$$

$$\pi_2 = \sqrt{J_1^2 - J_2^2}\cos\phi_2$$

$$\pi_3 = \sqrt{J_1^2 - J_2^2}\sin\phi_2 \tag{8}$$

which makes clear that the reduced phase space defined by π_0 = constant is a two-dimensional sphere. The action-angle variables just introduced are closely related to the coordinates of Hopf: transformations between the Hopf variables may be effected by a simple rotation on the Poincaré sphere.

Our intuition, which springs from molecular spectroscopy, causes us to think of the unperturbed Hamiltonian $\mathcal{H}_0$ as describing two modes (e.g., of a molecule) whose normal coordinates are taken to be x and y.[8] Working for now with the unperturbed Hamiltonian it is possible to characterize the invariant tori of the system in terms of the $\pi's$ or the quantities D, K, L. We label D-type dynamics *normal* mode, L-type dynamics *precessional* mode, and K-type dynamics *local* mode in analogy with the terminology used in molecular spectroscopy. In light of the Poisson bracket relations of eq. (4), π_0 and any *one* of D, K, and L (or linear combinations) may be used to parameterize the invariant tori of $\mathcal{H}_0$. The degeneracy of the unperturbed system means that any of the various possible representations are acceptable. Under the influence of a perturbation, however, it becomes critical to select the correct parametrization of the unperturbed tori. This must be done in such a way that, as one causes the perturbation to diminish, eventually to zero, the KAM tori of $\mathcal{H}$ transmute smoothly into the invariant tori of $\mathcal{H}_0$. In fact the three Hopf variables are associated with separability of the isotropic oscillator in Cartesian (π_1), polar (π_2) and rotated Cartesian coordinates (π_3). Importantly, rotations in the phase space coordinates transform the Hopf variables into each other, as can be proven by explicit calculation or by $SU(2)$ rotations on the Poincaré sphere.

Normalization of an elliptic oscillator perturbed by a real polynomial in the coordinates (x, y) about the center at the origin is guaranteed to produce an expansion in which each term is a polynomial in the $\pi's$. In the event that this polynomial is a function of a single Hopf variable the implication is that the dynamics of the reduced system is of purely local, normal or precessional mode nature. Does this imply anything regarding separability or integrability? It is time to normalize the exhibition problem.

Normal Form

Normalization of $\mathcal{H}$ produces the following expression through order ϵ^4 in terms of the Hopf variables

$$\begin{aligned}
\mathcal{H}_{NF} = 2\,\pi_0 + \epsilon^2 &\left(\frac{-15\,\alpha^2}{4} - \frac{5\alpha\beta}{2} - \frac{25\,\beta^2}{12}\right)\pi_0{}^2 \\
+\epsilon^2 \left(\frac{-15\,\alpha^2}{2} + \frac{5\,\beta^2}{6}\right)\pi_0\,\pi_1 &+ \epsilon^2\left(\frac{-15\,\alpha^2}{4} + \frac{5\alpha\beta}{2} + \frac{5\,\beta^2}{4}\right)\pi_1{}^2 \\
&+\epsilon^2(-\alpha\beta + 2\,\beta^2)\,\pi_2{}^2 \\
+\epsilon^4 &\left(\frac{-705\,\alpha^4}{16} - \frac{319\,\alpha^3\,\beta}{8} - \frac{151\,\alpha^2\,\beta^2}{8} - \frac{1427\,\alpha\,\beta^3}{72} - \frac{2887\,\beta^4}{432}\right)\pi_0{}^3 \\
+\epsilon^4 &\left(\frac{-2115\,\alpha^4}{16} - \frac{319\,\alpha^3\,\beta}{8} + \frac{59\,\alpha^2\,\beta^2}{12} - \frac{689\,\alpha\,\beta^3}{72} + \frac{2485\,\beta^4}{432}\right)\pi_0{}^2\,\pi_1 \\
+\epsilon^4 &\left(\frac{-2115\,\alpha^4}{16} + \frac{319\,\alpha^3\,\beta}{8} + \frac{151\,\alpha^2\,\beta^2}{8} + \frac{1091\,\alpha\,\beta^3}{72} + \frac{97\,\beta^4}{16}\right)\pi_0\,\pi_1{}^2 \\
+\epsilon^4 &\left(\frac{-705\,\alpha^4}{16} + \frac{319\,\alpha^3\,\beta}{8} - \frac{59\,\alpha^2\,\beta^2}{12} + \frac{1025\,\alpha\,\beta^3}{72} - \frac{739\,\beta^4}{144}\right)\pi_1{}^3 \\
+\epsilon^4 &\left(\frac{-41\,\alpha^3\,\beta}{4} - \frac{3\,\alpha^2\,\beta^2}{2} + \frac{767\,\alpha\,\beta^3}{36} + \frac{13\,\beta^4}{2}\right)\pi_0\,\pi_2{}^2 \\
+\epsilon^4 &\left(\frac{-41\,\alpha^3\,\beta}{4} + \frac{17\,\alpha^2\,\beta^2}{2} + \frac{545\,\alpha\,\beta^3}{36} - \frac{101\,\beta^4}{18}\right)\pi_1\,\pi_2{}^2
\end{aligned} \tag{9}$$

We now examine the normal form in the three known integrable limits.

(i) $\beta = 0$

$\mathcal{H}$ is clearly separable in the original Cartesian variables and the normal form in this limit, through 4^{th} order, and using eq. (5) becomes

$$\begin{aligned}
\mathcal{H}_{NF} = 2\,\pi_0 + \epsilon^2\alpha^2 &\left(\frac{-15\,\pi_0{}^2}{4} - \frac{15\,\pi_0\,\pi_1}{2} - \frac{15\,\pi_1{}^2}{4}\right) \\
+\epsilon^4\alpha^4 &\left(\frac{-705\,\pi_0{}^3}{16} - \frac{2115\,\pi_0{}^2\,\pi_1}{16} - \frac{2115\,\pi_0\,\pi_1{}^2}{16} - \frac{705\,\pi_1{}^3}{16}\right)
\end{aligned} \tag{10}$$

Notice that the normal form depends on the *single* Hopf variable π_1. We have verified that this pattern persists through 20^{th} order and we label it the normal mode limit. Importantly, π_1 is separable in the original coordinates and it is possible to express the normal form completely in terms of the two actions J_1 and J_2 of eq. (10) with no angle appearing. To us, dependence of the normal form on π_1 is the hallmark of separability in the original Cartesian coordinate system.

(ii) $\beta = 3\alpha$

In this case the normal form depends on a single Hopf generator, this time π_3, through at least 20^{th} order. Through 4^{th} order we obtain

$$\mathcal{H}_{NF} = 2\,\pi_0 - 15\,\alpha^2\,\epsilon^2\,\left(\pi_0{}^2 + \pi_3{}^2\right) - \frac{705}{2}\,\alpha^4\,\epsilon^4\,\pi_0\,\left(\pi_0{}^2 + 3\,\pi_3{}^2\right) \tag{11}$$

In this limit $\mathcal{H}$ is also separable after a rotation

$$\begin{aligned}
x &= (x' - y')/\sqrt{2}, \qquad & P_x &= (P'_x - P'_y)/\sqrt{2} \\
y &= (x' + y')/\sqrt{2}, \qquad & P_y &= (P'_x + P'_y)/\sqrt{2}
\end{aligned} \tag{12}$$

which, incidentally, has the effect of interchanging π_1 and π_3.

Taken together with results from previous studies of elliptic oscillators perturbed by quartic and sextic perturbations[4,5,7,8] it might seem that in the various integrable limits the normal form collapses into an expansion that involves a single Hopf variable. We have, however, discovered a counter example to this trend: $\mathcal{H}$ admits a third, and less well known integrable limit that corresponds to global separability in parabolic coordinates.

(iii) $\alpha = 2\beta$

Integrability in this limit was suggested by Chang, Tabor and Weiss[9] based on Painlevé analysis and numerical evidence. Motivated by this, J. Greene subsequently determined an analytic expression for the second invariant. This case is a particular limit of the more general Hamiltonian

$$\mathcal{H} = \frac{1}{2}(P_x^2 + P_y^2) + \frac{1}{2}(Ax^2 + By^2) + D(2x^3 + xy^2) \tag{13}$$

for which the second integral of the motion found by Greene is

$$I = D(yP_xP_y - xP_y^2) + y^2(D^2x^2 + D^2y^2/4 + DBx) + (B - A/4)(P_y^2 + By^2) \tag{14}$$

The Hamilton-Jacobi equation corresponding to eq. (14) is separable in shifted parabolic coordinates; for A = B = 1, and D = ϵ these coordinates are, $x = \sqrt{\xi\eta}$, $y = (\xi-\eta)/2 + 3/(4\epsilon)$.[10]. Examination of the 2^{nd} order normal form suggests that $\alpha = 2\beta$ might correspond to an integrable limit: substitution reduces the normal form at that order to an expression in terms of π_1. However, to higher order the normal form is the following

$$\begin{aligned}
\mathcal{H}_{NF} = 2\,\pi_0 - &\frac{5\,\epsilon^2\,\left(53\,\pi_0{}^2 + 70\,\pi_0\,\pi_1 + 21\,\pi_1{}^2\right)}{12} \\
&\frac{-5\,\epsilon^4\,\left(98999\,\pi_0{}^3 + 209755\,\pi_0{}^2\,\pi_1 + 145509\,\pi_0\,\pi_1{}^2\right)}{432} \\
&-\frac{5\,\epsilon^4\,\left(33033\,\pi_1{}^3 + 3360\,\pi_0\,\pi_2{}^2 + 2016\,\pi_1\,\pi_2{}^2\right)}{432}
\end{aligned} \tag{15}$$

which does not fit the pattern that might be expected, i.e., dependence on a single Hopf variable. It thus appears that the normal form is only sensitive to separability in the original Cartesian or polar coordinates in which case the associated separation constants are directly related to the Hopf generators. We now provide a second counter example from atomic physics in which normalization fails to uncover a non-separable integrable limit, but does detect all separable cases.

THE GENERALIZED VAN DER WAALS POTENTIAL

In Cartesian coordinates and atomic units ($m_e = e = \hbar = 1$) the Hamiltonian for the hydrogen atom in a GVDW potential is,[7,8,11]

$$H = \frac{1}{2}(P_x^2 + P_y^2 + P_z^2) + \frac{\gamma}{2}(x^2 + y^2 + \beta^2 z^2) - \frac{1}{r} \tag{16}$$

where E is energy and γ and β are dimensionless physical parameters. Scaling the coordinates by $\gamma^{-1/3}$ and the momenta by $\gamma^{1/6}$ and converting to cylindrical coordinates ($x = \rho\cos\phi, y = \rho\sin\phi, z$) gives the Hamiltonian,

$$\mathcal{H} = \gamma^{-1/3}H = \frac{1}{2}(P_\rho^2 + P_z^2) + \frac{1}{2}(\rho^2 + \beta^2 z^2) - \frac{1}{r} + \frac{m^2}{2\rho^2}. \qquad (17)$$

where $r = \sqrt{\rho^2 + z^2}$. In (17) m is the z-component of the angular momentum vector and is conserved due to axial symmetry.

The integrability of eq. (16) has been analyzed by a variety of methods, including Lie group and Painlevé methods. In particular, it has been shown that the equations of motion derived from (17) in semiparabolic coordinates with $m = 0$ have the Painlevé property for $\beta = \pm1, \pm2$, and $\pm1/2$.[11] Global invariants were obtained for these cases using Noether's theorem, but only for $m = 0$. A simpler and more direct resolution of the issue of the integrability of eq. (16) has recently been proposed; the invariants when $m = 0$ have been shown to be connected to separability of eq. (16) in appropriate coordinates.[12] However, the case $\beta = \pm1/2$ and arbitrary m is integrable but non-separable.

In order to normalize this problem we first pass over to the coordinates of Kustaanheimo and Stiefel (KS). The KS transformation has been described elsewhere in this connection[7] and results in a 4-dimensional 1:1:1:1 resonant Hamiltonian (after several scalings and rotations in phase space)

$$H_{KS} = \frac{4}{\omega} = \frac{1}{2}(\mathbf{P}_u^2 + |\mathbf{u}|^2) + \frac{8\epsilon^4(2-\beta^2)}{\omega^4}|\mathbf{u}|^2(u_1^2 + u_4^2)(u_2^2 + u_3^2)$$

$$+\frac{4\epsilon^4\beta^2}{\omega^4}|\mathbf{u}|^2\left[(u_1^2 + u_4^2)^2 + (u_2^2 + u_3^2)^2\right] \qquad (17)$$

where $\omega = \sqrt{-8E}$ which restricts eq. (17) to bound states with $E < 0$. The normal form was obtained in these coordinates using Mathematica as implemented on an IBM RISC 6000.[13] The normal form is most simply expressed in terms of the components of the orbital angular momenta, $\mathbf{L}(L_x, L_y, L_z)$ and the Runge-Lenz vector $\mathbf{A}(A_x, A_y, A_z)$ which generate the Lie algebra of the group $SO(4)$ [isomorphic to $SU(2) \otimes SU(2)$]. Using the usual notation for the generators of $SO(4)$, we write, $\mathbf{L} = \mathbf{L}(S_{23}, S_{13}, S_{12})$ and $\mathbf{A} = \mathbf{A}(S_{14}, S_{24}, S_{34})$. The generators of $SU(2) \otimes SU(2)$ are two angular momenta $\mathbf{J}$ and $\mathbf{K}$ which are related to the $SO(4)$ generators by,

$$\mathbf{J} = \frac{(\mathbf{L}+\mathbf{A})}{2}, \quad \mathbf{K} = \frac{(\mathbf{L}-\mathbf{A})}{2} \qquad (18)$$

where,

$$\{J_j, J_k\} = \epsilon_{jkl}J_l, \quad \{K_j, K_k\} = \epsilon_{jkl}K_l \qquad (19)$$

and $(i,j,k) = (x,y,z)$. The normal form through 4^{th} order is (where $n = \frac{1}{2}\mathcal{H}_0$)

$$\mathcal{H}_{NF} = 2n + 16\frac{\epsilon^4 n}{\omega^4}\left(n^2 + \beta^2 n^2 + {S_{12}}^2 - \beta^2\, {S_{12}}^2 + 4\,{S_{14}}^2 - \beta^2\, {S_{14}}^2\right)$$

$$+16\frac{\epsilon^4 n}{\omega^4}\left(4\,{S_{24}}^2 - \beta^2\, {S_{24}}^2 - {S_{34}}^2 + 4\beta^2\, {S_{34}}^2\right) \qquad (20)$$

and the resulting equations of motion for the generators are

$$\dot S_{12} = 0$$

$$\dot S_{13} = (1-\beta^2)\, S_{12}S_{23} - (4-\beta^2)\, S_{14}S_{34} + (4\beta^2-1)\, S_{14}S_{34}$$

$$\dot S_{14} = (1-\beta^2)\, S_{12}S_{24} - (4-\beta^2)\, S_{12}S_{24} - (4\beta^2-1)\, S_{13}S_{34}$$

$$\dot S_{23} = -(1-\beta^2)\, S_{12}S_{13} - (4-\beta^2)\, S_{24}S_{34} + (4\beta^2-1)\, S_{24}S_{34}$$

$$\dot S_{24} = -(1-\beta^2)\, S_{12}S_{14} + (4-\beta^2)\, S_{12}S_{14} - (4\beta^2-1)\, S_{23}S_{34}$$

$$\dot S_{34} = (4-\beta^2)\, S_{13}S_{14} + (4-\beta^2)\, S_{23}S_{24} \qquad (21)$$

Degenerate equilibria arise whenever the right hand sides of two or more of these equations vanish simultaneously. In fact, degenerate equilibria exist only in the three integrable limits. Further, this is true for all m except if $\beta = \pm1/2$ when degenerate equilibria exist only if $m = 0$. Hence, the cases producing degenerate equilibria coincide with the separable limits. Evidently, normalization has failed to detect the integrable limit $\beta = \pm1/2$, $|m| > 0$. Additional light is shed on the matter by examining the normal form itself in each integrable limit through higher order.

(i) $\beta = \pm2$

$$\mathcal{H}_{NF} = 2n + \frac{\epsilon^4\left(80n^3 - 48n\,{S_{12}}^2 + 240n\,{S_{34}}^2\right)}{\omega^4}$$

$$+\frac{\epsilon^8}{\omega^8}(-12576n^5 + 12480n^3\,{S_{12}}^2 - 2208n\,{S_{12}}^4$$

$$+\frac{\epsilon^8}{\omega^8}(-125760n^3\,{S_{34}}^2 + 37440n\,{S_{12}}^2\,{S_{34}}^2 - 62880n\,{S_{34}}^4) \qquad (22)$$

In this limit, through all computed orders, the normal form is found to depend only on the squares of the two generators $\{S_{12}, S_{34}\}$ *alias* $\{L_z, A_z\}$ whose Poisson brackets mutually vanish.

(ii) $\beta = \pm1$

This limit corresponds to rotational symmetry and the normal form becomes,

$$\mathcal{H}_{NF} = 2n + \frac{32\epsilon^4 n^3}{\omega^4} + \frac{48\epsilon^4 n}{\omega^4}(S_{14}^2 + S_{24}^2 + S_{34}^2) -$$

$$\frac{\epsilon^8 n^3}{\omega^8}\left({S_{14}}^2 + {S_{24}}^2 + {S_{34}}^2\right)\left(23\,{S_{14}}^2 + 23\,{S_{24}}^2 + 23\,{S_{34}}^2 + 84n^2\right) \qquad (23)$$

In view of the constraint $\mathbf{L.A} = 0$, pure dependence on $\mathbf{A}^2$ is equivalent to pure dependence on $\mathbf{L}^2$. This is expected, given the initial rotational invariance. Thus the reduced phase space is the sphere $SU(2)$ and the energies are degenerate with respect to m.

(iii) $\beta = \pm1/2$

$$\mathcal{H}_{NF} = 2n + \frac{20\epsilon^4 n^3}{\omega^4} + \frac{12\epsilon^4 n}{\omega^4}(m^2 + 5S_{14}^2 + 5S_{24}^2) -$$

$$6n\frac{\epsilon^8}{\omega^8}\left(23m^4 + 230m^2n^2 + 131n^4\right) +$$

$$\frac{30\epsilon^8 n}{\omega^8}\left(-62\,S_{14}{}^2 - 62S_{24}{}^2 + 16S_{34}{}^2\,m^2 - 131\,(S_{14}{}^2 + S_{24}{}^2)\,\,(S_{14}{}^2 + S_{24}{}^2 + 2n^2)\right) \tag{24}$$

Significantly, (S_{14}, S_{24}, S_{12}) generate an $SU(2)$ Lie algebra. Thus, through order ϵ^4 eq. (24) represents a dynamical symmetry limit of the original Hamiltonian because the normal form can be expressed entirely in terms of the dynamical quantities Λ^2 and Λ_3.. Therefore, the reduced phase space is the sphere $SU(2)$ defined by the components of Λ and a degenerate equilibrium exists. At orders higher than 4^{th}, terms in S_{34}^2 appear, but vanish in the limit $m = 0$. It is these terms that destroy any hope of finding degenerate equilibria when $m \neq 0$. Unlike the previous two examples, when $\beta = \pm 1/2$, for non-zero m the original Hamiltonian (16), while integrable, is non-separable and this correlates with the appearance of terms that break the dynamical symmetry. Incidentally, this result differs from Alhassid[14] who indicated that $\beta = 1/2$ is a dynamical symmetry limit for all m.

Normalization has a role to play in the detection of integrability. However, we have found two examples that clearly violate the conjecture that the normal form exhibits particular patterns that signify integrability or separability. We conclude that further study of the relationship of normal form theory to integrability would be profitable.

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EFFECTIVE STABILITY FOR PERIODICALLY PERTURBED HAMILTONIAN SYSTEMS

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Abstract

In this work we present a method to bound the diffusion near an elliptic equilibrium point of a periodically time-dependent Hamiltonian system. The method is based on the computation of the normal form (up to a certain degree) of that Hamiltonian, in order to obtain an adequate number of (approximate) first integrals of the motion. Then, bounding the variation of those integrals with respect to time provides estimates of the diffusion of the motion.

The example used to illustrate the method is the Elliptic Spatial Restricted Three Body Problem, in a neighbourhood of the points $L_{4,5}$. The mass parameter and the eccentricity are the ones corresponding to the Sun-Jupiter case.

1 Introduction

The study of the nonlinear stability of an elliptic equilibrium point of a Hamiltonian system is a classical and difficult topic. There are mainly two kind of results concerning this: results of KAM type (perpetual stability on a Cantor set of initial conditions) and results of Nekhoroshev type (stability for an exponentially long time span, on an open set of initial conditions). A survey of both kind of methods can be found in Arnol'd.[1]

In this work we are going to focus on the results of Nekhoroshev type. Our purpose will be to bound the diffusion of the motion near an elliptic equilibrium point of a Hamiltonian system. The kind of methods we are going to use is very similar to the ones used by Giorgilli et al.[2] and Simó[3] for autonomous Hamiltonians.

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