SPECTRAL ENERGY METHODS AND THE STABILITY OF SHOCK WAVES

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To Kristin

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Abstract

In this thesis, we review and motivate the stability problem for both viscous and relaxed shock waves, and discuss our recent work in proving spectral stability of small-amplitude shock profiles for physically realistic models, including gas dynamics and magnetohydrodynamics. Specifically, we use energy methods, extending the work of Goodman, Kawashima, Matsumura, and Nishihara, to prove spectral stability of small-amplitude shock profiles for the following one-dimensional systems: Kawashima class viscous conservation laws, the Jin-Xin relaxation model, and isentropic gas dynamics with capillarity.

The methods used herein are motivated by the above mentioned work, however, our analysis is carried out in the frequency domain, rather than the space-time domain. It has been recently shown, for rather general systems of these types, that spectral stability implies nonlinear stability. Thus, the stability problem for smooth shocks is reduced to determining the character of the spectrum of the linearized evolution operator, whereby spectral stability is the key.

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CHAPTER 1

Introduction

1. Motivation and background

In the study of hyperbolic conservation laws

$$u_t + f(u)_x = 0, \quad u, f \in \mathbb{R}^n, \tag{1.1}$$

one encounters discontinuous traveling wave solutions called *shock waves*. While some of these solutions are consistent with observed phenomena in nature, many are considered non-physical or mathematically spurious. As a result, additional criteria have been developed to determine admissibility. There are a host of important results in this direction [**47**].

As an alternative, Gelfand proposed his *vanishing viscosity method*, which is loosely based on the idea that the physically relevant solutions of (1) are expected to correspond to limiting solutions of viscous conservation laws

$$u_t + f(u)_x = \epsilon(B(u)u_x)_x, \quad u, f \in \mathbb{R}^n, \tag{1.2}$$

as $\epsilon \to 0$. These convection-diffusion models, when appropriately coupled, yield smooth solutions called *viscous shock waves*. This viewpoint lends additional analytical tools to the overall program of shock wave theory, but is met with some limits [**4, 5, 48**]. Several results demonstrate that the admissibility conditions determined by Gelfand's method are sensitive to choices of viscosity; whereby, different viscosities yield different limiting solutions of (1). It follows that the viscosity term in (2) plays an important role on the inner structure of shock waves and must be carefully chosen to be consistent with the physically relevant solutions.

Aside from being used to examine limiting solutions, viscous conservation laws are also fundamental in their own right. Many physical contexts, including gas dynamics, magnetohydrodynamics (MHD), and materials science, lead to viscous models – perhaps the most well-known example being the Navier-Stokes equations for compressible fluid dynamics.

In the context of viscous shocks, stability is a natural admissibility condition. As with any evolutionary system, some form of stability is what characterizes the homogeneity of observable phenomena, thus providing a notion of consistency in the light of experimental uncertainty. Viscous shock waves fall into a broader class of traveling wave phenomena, which are also found in reaction-diffusion equations, nonlinear optics, combustion models, etc. Many stability problems in these and other systems are mathematically similar to viscous shocks, and hence, several techniques can be applied broadly.

The general theme of this present work is shock wave stability. Zumbrun and his collaborators [**55, 20, 38, 39**], generalizing earlier work of [**41, 13, 34, 35, 49**] and others, have developed a general program for proving stability. In short, they have been able to show, for several general classes of shock waves, that spectral stability of the linearized operator implies nonlinear stability. Hence, the general stability problem for shocks is reduced to determining the character of the spectrum of the linearized operator. For spectral stability in this context, we mean that there do not exist any growth or oscillatory modes, i.e., no spectrum in the closed deleted half plane $\{\lambda \in \mathbb{C}\backslash\{0\} | \Re e(\lambda) \geq 0\}$.

2. Past Results

For viscous shock waves, we assume the *genuine coupling condition* – that there are no eigenvectors of $df(u)$ in the kernel of $B(u)$; otherwise solutions can decouple to form shocks, as in the hyperbolic case. To guarantee genuine coupling, one can impose a positive definite viscosity. While this restriction has substantial mathematical advantages, such as sectoriality in the semigroup framework and good uniform bounds for energy estimates, it is overly strong for many physical models. For example, both gas and plasma dynamics have models with degenerate viscosities due to strictly conserved quantities, which have no higher order terms in the presence of dissipation – e.g., *conservation of mass, charge, etc.* We note that the Navier-Stokes and MHD models have degenerate viscosities, yet satisfy the genuine coupling condition. Because of this, the restriction to the positive-definite case is often called *artificial viscosity*, where the general nonnegative-definite case, which allows for these degeneracies, is generally called *real viscosity*.

The ideas developed in the analysis of artificial viscosity are powerful and have driven much of the analysis in the real viscosity case. Majda and Pego [**37**] proved existence and uncovered the asymptotic structure for small-amplitude shocks. Using this structure, several results have been built up to achieve nonlinear stability for small-amplitude shocks. Goodman [**13**] used a clever weighted norm estimate to prove zero-mass stability[1]. T.-P Liu [**34**] provided partial results for the nonlinear stability problem, and Szepessy and Xin [**49**] later completed the proof. Finally, motivated by both Liu's work on pointwise Green's function bounds [**35**] and that of Gardner and Zumbrun [**10**] for their Evans function and "Gap Lemma" analysis, Zumbrun and Howard [**55**] proved that spectral stability implies nonlinear stability. This latest result holds for large-amplitude shocks as well.

[1]Zero-mass stability is defined as nonlinear stability subject to the a priori condition that admissible perturbations have integral zero. We remark that zero-mass stability is weaker than general nonlinear stability, but is slightly stronger than spectral stability.

The analysis for real viscosity has followed closely behind: Kawashima [**27**] developed a stability criterion, which combines genuine coupling with symmetrizability, that is, the condition that there exists a symmetric, positive definite matrix $A^0(u)$ such that $A^0(u)df(u)$ and $A^0(u)B(u)$ are both symmetric and $A^0(u)B(u)$ is non-negative definite. We call this the *Kawashima class*[2] and note that it contains the Navier-Stokes equations and the corresponding equations for MHD. This is viewed as a structural stability condition, which implies stability of constant solutions in L^2. Later, Pego [**42**] proved existence for small-amplitude shocks by extending the work of [**37**], stated above, to the Kawashima class.

The stability problem for conservation laws with real viscosity was initially examined by Kawashima, Matsumura, and Nishihara [**41, 28, 29**], who proved zero-mass stability of small-amplitude shocks for γ-law gas dynamics. Their methods involve clever energy estimates which capitalize on the structure of the equations. While they solved an important problem, it does not appear that their technique extends to the rest of the Kawashima class.

3. Recent Results

3.1. Stability of Kawashima Class Viscous Shocks. In this present work (see also [**23**]), we use a series of energy estimates to prove spectral stability for small-amplitude Kawashima class shock waves. Our approach extends both the work of Goodman for the case of artificial viscosity and that of Kawashima for real viscosity, see Chapter 3.

An interesting remark is that all of the above estimates were carried out in the frequency domain as spectral energy estimates, rather than the more traditional time-asymptotic estimates. In addition, there does not appear to be an obvious

[2]The Kawashima class is actually more general than this. See Appendix A for details.

time-asymptotic analog to our work. This approach was inspired by Zumbrun and Howard's program. It is worth pointing out that while spectral stability for γ-law gas dynamics was already proven (mentioned above in [**41, 29, 28**]), the stability of small-amplitude shocks in MHD was not known, and is implied by our work.

Finally we mention that Mascia and Zumbrun [**38**] generalized the work of Zumbrun and Howard, by showing that spectral stability implies nonlinear stability for the Kawashima class. Thus, combined with our spectral stability result, the stability problem for small-amplitude Kawashima class viscous shocks is complete.

3.2. Stability of Relaxation Shocks. Aside from viscous conservation laws, the stability of shock waves in relaxation models is also of great interest. Relaxation shows up in several physical situations, in particular, the kinetic theory of gases. T.-P Liu [**34**] proved zero-mass stability for small-amplitude shocks in the general 2×2 relaxation model

$$\begin{aligned} &u_t + f(u,v)_x = 0, \\ &v_t + g(u,v)_x = h(u,v), \quad u, v \in \mathbb{R}, h_v < 0. \end{aligned} \tag{1.3}$$

Later, Caflisch and Liu [**3**] proved the same for the Broadwell model, which is a 3×3 relaxation system. In this work we consider the Jin-Xin relaxation model, which is a popular $2n \times 2n$ system used in numerical studies of hyperbolic equations and is included in the general form of the Kawashima class. We show that Jin-Xin relaxation shocks are spectrally stable in Chapter 4 (see also [**21**]).

Mascia and Zumbrun [**39**] have further extended the work of Zumbrun and Howard to include relaxation models. Hence, for the Jin-Xin model, as well as the other models mentioned above, the stability problem for small-amplitude relaxation shocks is now complete.

Author's Note: At the time of this writing, Plaza and Zumbrun proved spectral stability for general small-amplitude relaxation shocks [**43**].

3.3. Stability of Viscous-Dispersive Shocks. Systems with added dispersion are also very interesting. In Chapter 5 (see also [**22**]), we examine stability problem for isentropic gas dynamics with capillarity [3]

$$
\begin{aligned}
&v_t - u_x = 0,\\
&u_t + p(v)_x = (b(v)u_x)_x + dv_{xxx}, \quad u, v \in \mathbb{R},
\end{aligned}
\tag{1.4}
$$

where $p'(v) < 0$, $p''(v) > 0$, $d < 0$. We prove that small-amplitude shocks are spectrally stable. In addition we prove that there are no positive real eigenvalues for monotone profiles in this model – a result that is independent of the shock amplitude.

3.4. Methodology. Our approach in this work makes use of spectral energy estimates; meaning that we perform energy estimates on the eigenvalue problem determined by linearizing about the profile. Traditionally, energy estimates are more commonly done with the evolution equations. However, our approach is motivated by the idea that spectral estimates can have advantages over the time-asymptotic approach. Specifically, as we see in Chapter 3, one can sometimes combine estimates in a straightforward manner, which would be difficult at best in the traditional approach. In any case, energy estimates require that appropriate weights be applied to leverage the special structure enough for the "good terms" to dominate the "error terms". We remark that for the systems that we consider herein, small-amplitude shock wave profiles have small derivatives. This is a key fact that is used throughout our analysis by making our higher order error terms relatively small in our estimates. In the large-amplitude case, our stability proofs do not hold. Indeed there only two known results in the literature for large-amplitude shocks [**41, 54**], both of which have very special structure.

[3] A similar problem to the one treated here has been examined by Kodja [**31**].

CHAPTER 2

Mathematical Background

In this chapter, we depart momentarily from the theory of shock waves, per se, and consider a broader class of equations that admit more general traveling wave solutions.

1. Traveling waves

We consider the following class of quasi-linear systems of partial differential equations (PDE)

$$u_t + f(u)_x - (B(u)u_x)_x + (C(u)u_{xx})_x + Q(u) = 0, \tag{2.1}$$

where $x \in \mathbb{R}$, $u, f \in \mathbb{R}^n$, and $B, C, Q \in \mathbb{R}^{n\times n}$ are all twice continuously differentiable.

Definition 2.1. *A traveling wave profile of* (2.1) *is a solution $\hat{u}$ satisfying*

$$u(x,t) = \hat{u}(x - st), \tag{2.2}$$

where s is the speed of the wave. By assuming a solution of the form (2.2)*, the existence problem for a traveling wave profile reduces to the ordinary differential equation (ODE),*

$$(f'(u) - s)u' - (B(u)u')' + (C(u)u'')' + Q(u) = 0, \tag{2.3}$$

subject to appropriate boundary conditions.

Different boundary conditions lead to different classes of traveling waves. In this work, we restrict ourselves to the class of continuous traveling waves with asymptotically constant boundary, i.e., those traveling waves satisfying

$$\lim_{x\to\pm\infty} \hat{u}(x) = u_\pm \quad \text{and} \quad \lim_{x\to\pm\infty} \hat{u}^{(n)}(x) = 0, \quad n \geq 1. \tag{2.4}$$

REMARK 2.1. *Notice that the endpoints $u_\pm$ are equilibrium points for* (2.3)*, when considered as a dynamical system. Hence, we see that traveling waves correspond to connecting orbits in phase space. In fact, since* (2.3) *is autonomous in x, the invariance under translations, $x \to x + \delta$, yields a smooth one-parameter manifold $\{\hat{u}^\delta\}$ of traveling wave profiles corresponding to a single connecting orbit.*

DEFINITION 2.2. *Let $\hat{u}$ be a nontrivial traveling wave profile satisfying* (2.4)*. If $u_- \neq u_+$, then $\hat{u}$ is called a wave front with amplitude $\epsilon = |u_+ - u_-|$. If $u_- = u_+$, then $\hat{u}$ is called a pulse.*

REMARK 2.2. *In the context of systems that smoothly regularize hyperbolic models, front waves are generally referred to as shock waves, despite the fact that they are smooth traveling waves.*

1.1. Examples. Traveling waves are found pervasively throughout the mathematical sciences. Such examples are found in areas of continuum mechanics, kinetic theory, biological applications, chemical reactions, materials science, combustion theory, and the study of phase transitions. We devote the remainder of this section to providing examples of traveling waves.

EXAMPLE 2.1 (Viscous Burgers equation). *We consider the scalar convection-diffusion equation*

$$u_t + uu_x = \nu u_{xx}, \quad \nu > 0. \tag{2.5}$$

This well-known model characterizes behavior found in shock tube experiments. The shock wave solutions of (2.5) *satisfy the following boundary value problem*

$$-su' + uu' = \nu u'',$$

$$u(\pm\infty) = u_{\pm}.$$

By integrating (2.1) *from* x *to* $-\infty$ *and simplifying, we reduce to*

$$u^2 - 2s(u - u_-) - u_-^2 = 2\nu u'.$$

By completing the square and integrating, we find the exact solution,

$$u(x,t) = s - a \ tanh\left[\frac{a(x - st + \delta)}{2\nu}\right],$$

where $a = (u_- - u_+)/2$ *and* $\delta \in \mathbb{R}$ *is some translate.*

Remark 2.3. *Notice that in the limit as* $\nu \to 0$*, the viscous Burgers solution approximates a genuine shock wave solution as prescribed from hyperbolic theory.*

Example 2.2 (Viscous conservation law). *This general class includes: the viscous Burgers equation above, the celebrated Navier-Stokes equation, and magnetohydrodynamics (MHD). It satisfies* (2.1) *when* $Q = C = 0$.

$$u_t + f(u)_x = (B(u)u_x)_x, \tag{2.6}$$

The traveling wave ODE is given by

$$(f(u) - s)u_x = (B(u)u_x)_x, \tag{2.7}$$

subject to (2.4)*. By integrating* (2.7) *from* x *to* $-\infty$*, we arrive at*

$$B(u)u_x = -s(u - u_-) + (f(u) - f(u_-)). \tag{2.8}$$

Example 2.3 (Jin-Xin relaxation [**24**]). *Consider the following model, which has roots in kinetic theory,*

$$\begin{aligned} U_t + V_x &= 0, \\ V_t + AU_x &= f(U) - V, \end{aligned} \tag{2.9}$$

which satisfies (2.1) *when* $B = C = 0$*. The traveling wave (shock wave) profiles are given by the ODE*

$$\begin{aligned} -sU' + V' &= 0, \\ -sV' + AU' &= f(U) - V, \end{aligned} \tag{2.10}$$

subject to (2.4)*. By combining terms and integrating, we reduce the traveling wave ODE to*

$$(A - s^2 I)U' = -s(U - U_-) + f(U) - V_-. \tag{2.11}$$

Since $U' = 0$ *on the boundary, it follows that* $f(U_-) = V_-$*. Hence*

$$(A - s^2 I)U' = -s(U - U_-) + f(U) - f(U_-), \tag{2.12}$$

which is the same ODE as (2.8)*, when* $B = A - s^2 I$*, and thus the respective profiles are the same.*

Example 2.4 (Korteweg-deVries equation). *We consider the convection-dispersion model, which is used to study, among other things, shallow wave motion*

$$u_t + 6uu_x + u_{xxx} = 0. \tag{2.13}$$

Notice that this satisfies (2.1)*, when* $f(u) = 3u^2$*,* $C(u) = 1$*, and* $B(u) = Q(u) = 0$*. We seek pulse solutions for this model. The traveling wave ODE is given by*

$$-su_x + 6uu_x + u_{xxx} = 0, \tag{2.14}$$

subject to the condition that $u_+ = u_- = 0$*. We can integrate to get*

$$-su + 3u^2 + u_{xx} = 0.$$

By multiplying through by u_x *and integrating again, we arrive at*

$$u_x^2 = su^2 - 2u^3,$$

which has the exact solution

$$u(x,t) = \frac{s}{2} sech^2 \left[\frac{\sqrt{s}}{2}(x - st + \delta) \right],$$

where $\delta \in \mathbb{R}$ *is a translate. Notice that the amplitude is dependent on the speed of the pulse. This is a general characteristic of solitary waves.*

EXAMPLE 2.5 (Scalar reaction-diffusion equation). *We briefly examine the general reaction-diffusion model, which has numerous chemical and biological applications,*

$$u_t + F(u) = B(u)u_{xx}. \tag{2.15}$$

This satisfies (2.1) *when* $f(u) = C(u) = 0$ *and* $Q(u) = F(u)$*. The traveling wave ODE takes the form*

$$B(u)u'' + su' - F(u) = 0. \tag{2.16}$$

2. Stability of traveling waves

In the previous section, we examined the existence problem for traveling waves. Indeed we showed, for the class of traveling waves with asymptotically constant boundary, that the existence problem reduces to finding a connecting orbit in phase space. From a more general viewpoint, we can consider traveling waves as stationary solutions in a moving frame. Specifically, we can arrive at (2.3) by translating equation (2.1) via $(x,t) \to (x - st, t)$ and considering the stationary solutions of

$$u_t = \mathcal{F}(u) = -(f'(u) - s)u_x + (B(u)u_x)_x - (C(u)u_{xx})_x - Q(u). \tag{2.17}$$

This latter point of view allows us to consider a traveling wave profile in the context of a general evolutionary system of the form $u_t = \mathcal{F}(u)$, where $u(\cdot,t)$ is in an appropriate Banach space $\mathcal{X}$. In this setting, a traveling wave profile is a stationary solution $\hat{u}$ of (2.17), i.e., satisfies $\mathcal{F}(\hat{u}) = 0$. By characterizing traveling wave profiles in this manner, we can examine the stability problem by considering the long-term behavior of solutions which are *initially* "close to" the equilibrium solution $\hat{u}$.

2.1. Orbital Stability. Given an appropriate Banach space $\mathcal{X}$ with norm $\|\cdot\|$ and an admissible set $\mathcal{A} \subset \mathcal{X}$ of small perturbations, we can consider Cauchy problem for (2.17) with initial data

$$u(x,0) = \hat{u}(x) + v(x,0), \quad v(x,0) \in \mathcal{A},$$

where $\hat{u}$ is a stationary solution of (2.17). Note that $\|u(x,0) - \hat{u}(x)\| = \|v(x,0)\|$. Hence, the evolution of $v(\cdot,t)$ describes the difference in behavior between the stationary solution $\hat{u}$ and $u(x,t)$.

By linearizing (2.17), i.e., expanding out the linear and nonlinear parts, we have that $v(x,t)$ satisfies

$$v_t = \underbrace{-(Av)_x + (Bv_x)_x - (Cv_{xx})_x - Dv}_{\text{Linear term}=L(v)} + \underbrace{R(v, v_x, v_{xx})_x + S(v)}_{\text{Higher order}} \tag{2.18}$$

where $R = \mathcal{O}(|v|^2 + |v'|^2 + |v''|^2)$, $S = \mathcal{O}(|v|^2)$ and

$$\begin{aligned} &Av := df(\hat{u})v - dB(\hat{u})v\hat{u}_x + dC(\hat{u})v\hat{u}_{xx}, \\ &B := B(\hat{u}), \quad C := C(\hat{u}), \quad \text{and} \quad D := dQ(\hat{u}). \end{aligned} \tag{2.19}$$

By Duhamel's principle, we express the solution of $v(x,t)$ as the sum of a linear term and a nonlinear term

$$v(x,t) = e^{Lt}v(x,0) + \int_0^t e^{L(t-s)}(R(v, v_x, v_{xx})_x + S(v))\, ds. \tag{2.20}$$

Formally we can see that asymptotic stability occurs when $v(x,t) \to 0$ as $t \to \infty$. However, since our equilibrium solution $\hat{u}$ is just one point of a continuous manifold of equilibrium solutions $\{\hat{u}^\delta\}$, the best we can generally hope for is orbital stability. We have the following definition:

Definition 2.3. *A stationary solution $\hat{u}$ of (2.17) is (asymptotically) orbitally stable with respect to $\mathcal{A}$ if $u(\cdot,t) \to \{\hat{u}^\delta\}$ as $t \to \infty$, whenever $u(\cdot,0) - \hat{u} \in \mathcal{A}$. We use the terms orbital stability and nonlinear stability interchangeably.*

We seek a general program for determining whether a given traveling wave profile is stable. Motivated by Lyapunov's theorem for hyperbolic equilibrium points in dynamical systems theory, it is desirable to develop a stability theory for traveling waves, which equates orbital stability with a "well-behaved" spectrum. There are several difficulties toward this end. Before expounding on this further, however, we first describe the spectral problem for the linearized operator.

The spectrum of a traveling wave profile comes from the eigenvalue problem associated with the linearization (2.18) of the evolution operator in (2.17)

$$\lambda v = Lv := -(Av)_x + (Bv_x)_x - (Cv_{xx})_x - Dv. \tag{2.21}$$

DEFINITION 2.4. *We have the following:*

(i). *The spectrum $\sigma(L)$ of L is the set of all $\lambda \in \mathbb{C}$ such that $L - \lambda I$ is not invertible, i.e., there does not exist a bounded inverse.*

(ii). *The point spectrum $\sigma_p(L)$ of L is the set of all isolated eigenvalues of L with finite multiplicity.*

(iii). *The essential spectrum $\sigma_e(L)$ of L is the entire spectrum less the point spectrum, i.e., $\sigma_e(L) = \sigma(L) \backslash \sigma_p(L)$.*

In the following lemma, we show that L always has at least one eigenvalue:

LEMMA 2.1 (Sattinger [**44**]). *The derivative of the profile $\hat{u}'$ is an eigenfunction of L with eigenvalue* 0.

PROOF. By the translational invariance discussed in Remark 2.1, $\mathcal{F}(\hat{u}(x+\delta)) = 0$, for all $\delta \in \mathbb{R}$. Hence, differentiating with respect to δ and evaluating at $\delta = 0$, yields $d\mathcal{F}(\hat{u})\hat{u}' = L\hat{u}' = 0$. □

Definition 2.5. *We say that the operator L in* (2.21)*, linearized about the profile $\bar{u}(\cdot)$, is spectrally stable if there is no spectrum in the closed deleted right half-plane*

$$\Sigma_+ = \{\lambda \in \mathbb{C}\backslash\{0\} | \, \Re e\lambda \geq 0\}. \tag{2.22}$$

3. Spectral stability

To prove spectral stability, we must exclude both essential and point spectrum from Σ_+. We break the problem into two parts:

3.1. Essential Spectrum. It turns out that excluding the essential spectrum is relatively easy, due to the following theorem:

Theorem 2.1 (Henry [**16**]). *The essential spectrum of L in* (2.21) *is sharply bounded to the left of*

$$\sigma_e(L_+) \cup \sigma_e(L_-), \tag{2.23}$$

where $L_\pm$ correspond to the operators obtained by linearizing about the constant solutions $\hat{u} = u_\pm$, respectively.

Linearizing about constant solutions $u_\pm$ gives us the linear PDE

$$v_t = L_\pm v = -A_\pm v_x + B_\pm v_{xx} - C_\pm v_{xxx} - D_\pm v, \tag{2.24}$$

where $A_\pm, B_\pm, C_\pm, D_\pm$ in (2.19) are all constant matrices. We note that constant coefficient linear operators have no point spectrum and hence $\sigma(L_\pm) = \sigma_e(L_\pm)$. Formally, we can determine $\sigma_e(L_\pm)$ by considering the Fourier transform. Note that

$$(\widehat{L - \lambda I})^{-1} v = (-i\xi A_\pm - \xi^2 B_\pm + i\xi^3 C_\pm - D_\pm - \lambda I)^{-1} v, \quad \xi \in \mathbb{R}. \tag{2.25}$$

We lose invertibility of $L - \lambda I$ when $-i\xi A_\pm - \xi^2 B_\pm + i\xi^3 C_\pm - D_\pm - \lambda I$ is singular. Thus we can see that

$$\lambda \in \sigma(L_\pm) \quad \text{iff} \quad \lambda \in \sigma(-i\xi A_\pm - \xi^2 B_\pm + i\xi^3 C_\pm - D_\pm), \tag{2.26}$$

for some $\xi \in \mathbb{R}$. This defines $2n$-curves $\lambda_j^{\pm}(\xi)$ corresponding the eigenvalues of the right-hand side. Thus, we have

$$\sigma_e(L_+) \cup \sigma_e(L_-) = \bigcup_j \lambda_j^+(\xi) \cup \bigcup_j \lambda_j^-(\xi). \tag{2.27}$$

To summarize, the structure of the constant matrices $A_\pm, B_\pm, C_\pm, D_\pm$ determines sharp bounds on the essential spectrum of L. Hence, we can explicitly compute $\sigma_e(L)$ and check if it intersects Σ_+.

3.2. Point Spectrum. Computing bounds on the point spectrum is much more difficult than the essential spectrum. Probably the best known and historically successful technique to do this is to use energy estimates. The idea is to leverage the structure of the given equations. This is done, generally, by finding a weighted norm that is sufficiently dominant in some sense that it can absorb error terms. These techniques are generally very specialized to the particular system being considered and often appear to be somewhat mysterious and nonintuitive.

Additionally for traveling waves, it is difficult to find uniform bounds in energy estimates which will exclude Σ_+ while also allowing for zero. One remedy, for the (reactionless) subclass of (2.1),

$$\lambda v = Lv = -(Av)' + (Bv')' - (Cv'')', \tag{2.28}$$

is to consider instead the "integrated operator" (see [**14, 15, 41, 28, 29**])

$$\lambda w = \mathcal{L}w = -Aw' + Bw'' - Dw'''. \tag{2.29}$$

As we see in the following lemma, the point spectrum between L and $\mathcal{L}$ differ only at zero, and hence one is spectrally stable if and only if the other is.

Lemma 2.2. *The point spectrum of the original operator L and that of the "integrated operator" $\mathcal{L}$ agree everywhere except at zero. In particular they agree on the unstable half-plane Σ_+.*

PROOF. Let v satisfy $\lambda v = Lv$, for $\lambda \neq 0$. Define

$$w(x) := \int_{-\infty}^{x} v(y)dy. \tag{2.30}$$

By integrating (2.28) we can see that w and its derivatives decay to zero at $\pm\infty$, i.e.,

$$\lambda w(+\infty) = \lambda \int_{-\infty}^{+\infty} v = -\int_{-\infty}^{+\infty} (Av)' + \int_{-\infty}^{+\infty} (Bv')' - \int_{-\infty}^{+\infty} (Cv'')' = 0.$$

By substituting w' for v in (2.28) and integrating from x to $-\infty$, we arrive at (2.29). Hence $\sigma_p(L)\backslash\{0\} \subset \sigma_p(\mathcal{L})$. Conversely, let w satisfy $\lambda w = \mathcal{L}w$, for $\lambda \neq 0$. We use (2.30) to substitute v for w in (2.29). We have

$$\lambda \int_{-\infty}^{x} v(y) = -Av + Bv' - Dv''.$$

Differentiating gives (2.28). Hence $\sigma_p(\mathcal{L}) \subset \sigma_p(L)$. $\square$

3.3. Scalar Conservation Law. We conclude this section by proving spectral stability for the scalar conservation law

$$u_t + f(u)_x = (b(u)u_x)_x,$$

where $f'(u) < 0$, $f''(u) > 0$, and $b(u) > 0$. According to Example 2.2, viscous shocks satisfy the scalar ODE

$$u_x = \frac{1}{b(u)}\left[-s(u-u_-) + (f(u) - f(u_-))\right].$$

Without loss of generality, assume $u_- > u_+$, i.e., $\hat{u}_x < 0$. By linearizing about the profile we get the following eigenvalue problem:

$$\lambda v = \mathcal{L}v = -\left[(f'(\hat{u}) - b'(\hat{u})\hat{u}_x)v\right]' + (b(v)v')'.$$

We exclude the essential spectrum by taking the Fourier transform of $L_\pm$

$$\lambda v = \widehat{L_\pm v} = -i\xi a_\pm v - \xi^2 b_\pm v,$$

where $a_\pm = f'(\hat{u}_\pm)$. Taking the inner product with v yields

$$\lambda\langle v, v\rangle = -i\xi a_\pm\langle v, v\rangle - \xi^2 b_\pm\langle v, v\rangle. \tag{2.31}$$

Hence,

$$\lambda(\xi) = -i\xi a_{\pm} - \xi^2 b_{\pm},$$

which corresponds to parabolic curves in the left-half plane of $\mathbb{C}$ centered at $z = 0$. Alternatively, by taking the real part of (2.31), we can see that $\Re e \lambda(\xi) = -\xi^2 b_{\pm} < 0$ for all $\xi \neq 0$. Hence the essential spectrum does not intersect Σ_+.

We now examine the point spectrum by transforming the eigenvalue problem into the integrated coordinate

$$\lambda v = \mathcal{L} v = -(f'(\hat{u}) - b'(\hat{u})\hat{u}_x)v' + b(v)v''.$$

We take the L^2 inner product with v

$$\lambda \int_{-\infty}^{+\infty} |v|^2 = -\int_{-\infty}^{+\infty} (f'(\hat{u}) - b'(\hat{u})\hat{u}_x)v'\bar{v} + \int_{-\infty}^{+\infty} b(\hat{u})v''\bar{v}.$$

Integrate the last term by parts

$$\lambda \int_{-\infty}^{+\infty} |v|^2 = -\int_{-\infty}^{+\infty} (f'(\hat{u}) - b'(\hat{u})\hat{u}_x)v'\bar{v} - \int_{-\infty}^{+\infty} b'(\hat{u})\hat{u}_x v'\bar{v} - \int_{-\infty}^{+\infty} b(\hat{u})|v'|^2,$$

which simplifies to

$$\lambda \int_{-\infty}^{+\infty} |v|^2 + \int_{-\infty}^{+\infty} f'(\hat{u})v'\bar{v} + \int_{-\infty}^{+\infty} b(\hat{u})|v'|^2 = 0.$$

Thus, we take the real part

$$\Re e(\lambda) \int_{-\infty}^{+\infty} |v|^2 - \frac{1}{2}\int_{-\infty}^{+\infty} f''(\hat{u})\hat{u}_x |v|^2 + \int_{-\infty}^{+\infty} b(\hat{u})|v'|^2 = 0.$$

This is a contradiction for $\Re e(\lambda) \geq 0$ since all the terms on the left-hand side are positive. Hence, shock waves in scalar conservation laws are spectrally stable.

CHAPTER 3

Stability of viscous shocks

1. Introduction

Consider a one-dimensional system of conservation laws

$$u_t + f(u)_x = (B(u)u_x)_x, \tag{3.1}$$

where $u,\ f \in \mathbb{R}^n$, $B \in \mathbb{R}^{n\times n}$, $f,\ B \in C^2$, that is of *symmetric hyperbolic–parabolic type* in the sense of Kawashima[1] [**27, 30**] in some neighborhood $\mathcal{U}$ of a particular base point u_*, i.e., the following assumptions hold:

Assumption 3.1 (Symmetrizability). *For all $u \in \mathcal{U}$, there exists a symmetrizer $A^0(u)$, symmetric and positive definite, such that both $A^0(u)df(u)$ and $A^0(u)B(u)$ are symmetric, and $A^0(u)B(u)$ is nonnegative definite.*

Assumption 3.2 (Genuine coupling). *For $u \in \mathcal{U}$, there is no eigenvector of $df(u)$ lying in the kernel of $B(u)$.*

Assumption 3.3 (Block structure). *The left kernel of $B(u)$ is independent of u. (Author's note: In* [**23**] *we used the assumption that the right kernel of $B(u)$ is independent of u. While it does not matter for the linearized analysis contained herein, the nonlinear analysis in* [**39**] *uses the left block structure, and so we make that assumption here.)*

[1] For a more general description of the Kawashima class, see Appendix A.

These properties are enjoyed by many of the equations of continuum mechanics, in particular the equations of compressible fluid dynamics and magnetohydrodynamics. In such applications, the kernel of $B(u)$ is generally nontrivial. This observation corresponds to strictly conserved quantities, e.g., mass, charge, etc., which remain hyperbolic, despite higher order effects such as dissipation. The significance of Assumptions 3.1–3.3 is that behavior is nonetheless similar in many ways to what would be seen in the strictly parabolic case. For example, the "genuine coupling" of hyperbolic and parabolic effects embodied in Assumption 3.2 has been shown in several contexts to imply time-asymptotic smoothing and large-time behavior similar to that of the strictly parabolic case [**40, 26, 51, 36, 17, 18, 19**].

In particular, at least for small-amplitude waves, conditions Assumptions 3.1–3.3 imply that the viscosity B is sufficiently regularizing to "smooth" discontinuous traveling wave solutions, or "shock waves,"

$$u(x,t) = \hat{u}(x-st) := \begin{cases} u_- & x-st<0, \\ u_+ & x-st \geq 0, \end{cases} \tag{3.2}$$

of the corresponding hyperbolic equations

$$u_t + f(u)_x = 0, \tag{3.3}$$

yielding instead *smooth* traveling wave solutions

$$u = \hat{u}(x-st); \qquad \lim_{z\to\pm\infty} \hat{u}(z) = u_\pm, \tag{3.4}$$

or "viscous shock profiles". This fact is well-known in the context of gas dynamics [**50, 12**], and was established by Pego [**42**].

More precisely, let

$$a_1(u) \leq \cdots \leq a_n(u) \tag{3.5}$$

denote the eigenvalues of $A := df(u)$, $r_j(u)$ and $l_j(u)$ a smooth choice of associated right and left eigenvectors, $l_j \cdot r_k = \delta^j_k$, and assume at the base point u_* that the following assumptions hold:

Assumption 3.4 (Simplicity). *The pth characteristic field is of multiplicity one, i.e. $a_p(u_*)$ is a simple eigenvalue of $A(u_*)$.*

Assumption 3.5 (Genuine nonlinearity). *The pth characteristic field is genuinely nonlinear, i.e. $\nabla a_p \cdot r_p(u_*) \neq 0$.*

Remark 3.1. *We note that Assumption 3.5 is not needed either for the existence or the stability result, but is made only to simplify the discussion. Existence was treated for the general (nongenuinely nonlinear) case in* [**42**]. *Likewise, to extend our stability argument to the general case, one has only to substitute for the "Goodman-type" weighted energy estimate in Section 6, the variation introduced by Fries* [**9, 8**] *to treat the nongenuinely nonlinear case for strictly parabolic viscosities; for, at this point in the argument, the situation is reduced essentially to that of the strictly parabolic case. We suspect, further, that $\Re e A^0 B \geq 0$ can be substituted in Assumption 3.1 for the symmetric, nonnegative definite assumption on $A^0 B$, in both the existence and stability theory, with little change in the arguments.*

In this chapter, we show that small-amplitude Kawashima class viscous shocks are spectrally stable. This result may be viewed as a generalization of the zero-mass results obtained early on by Matsumura–Nishihara [**41**] and Kawashima–Matsumura–Nishihara [**28, 29**] for small-amplitude shocks of the equations of compressible gas dynamics. It can also be viewed as a generalization of the corresponding result of Goodman [**14, 15**] for small-amplitude shocks of general, strictly parabolic systems, which appeared at roughly the same time.

Interestingly, these two apparently similar results proceed by rather different arguments. Indeed, though it seems natural to conjecture that the results of [**41, 28, 29**] should extend to general Kawashima class systems, we do not see an obvious way to extend the approach of [**41, 28, 29**] to more general systems. We proceed here, instead, by adapting the weighted energy method of Goodman [**15**] to the degenerate viscosity case, thus achieving a unified approach to the degenerate and the strictly parabolic viscosity case.

The structure of our argument is straightforward: Since Goodman's approach involves coordinate changes not respecting the spectral structure of A^0B, the resulting diffusion term may in fact be *indefinite*, yielding unfavorable energy estimates in certain modes. However, the extent of deviation from semidefinite positivity is small on the order of the shock amplitude, and so the resulting bad H^1 term in the energy estimate can be controlled by higher order energy estimates of the type described by Kawashima [**27**]. An interesting aspect of the analysis is that here, in contrast to [**27**], the approach of Kawashima is applied to perturbations of a *nonconstant* background solution, confirming the flexibility of the method. The following theorem is key to our derivative estimate and our ability to exclude the essential spectrum:

Theorem 3.1 (Shizuta and Kawashima [**45**]). *Given Assumption 3.1, we have that Assumption 3.2 is equivalent to either of:*

(i). *For each $u \in \mathcal{U}$, there exists a skew-symmetric matrix $K(u)$ such that*

$$\Re e(KA^0A + A^0B)(u) \geq \theta > 0, \tag{3.6}$$

where A^0 is as in Assumption 3.1.

(ii). *For some $\theta > 0$, there holds*

$$\Re e\left[\sigma(-i\xi A(u) - |\xi|^2 B(u))\right] \leq -\theta|\xi|^2/(1+|\xi|^2), \tag{3.7}$$

for all $\xi \in \mathbb{R}$.

PROOF. For the proof of these and other useful equivalent formulations of Assumption 3.2, see Appendix A. □

2. Preliminaries

2.1. Existence and asymptotics.

PROPOSITION 3.1 (Pego [**42**]). *Let Assumptions 3.1–3.5 hold. Then, for left and right states $u_\pm$ lying within a sufficiently small neighborhood $\mathcal{V} \subset \mathcal{U}$ of u_*, and speeds s lying within a sufficiently small neighborhood of $a_p(u_*)$, there exists a viscous profile* (3.4) *that is "local" in the sense that the image of $\hat{u}(\cdot)$ lies entirely within $\mathcal{V}$ if and only if the triple (u_-, u_+, s) satisfies both the Rankine–Hugoniot relations:*

$$s[u] = [f], \tag{3.8}$$

and the Lax characteristic conditions for a p-shock:

$$a_p(u_-) > s > a_p(u_+); \quad \mathrm{sgn}(a_j(u_-) - s) = \mathrm{sgn}(a_j(u_+) - s) \neq 0 \textit{ for } j \neq p.$$

REMARK 3.2. *The structure theorem of Lax* [**32, 47**] *implies that* (3.8)*, always a necessary condition for existence of profiles, holds for $u_\pm \in \mathcal{V}$ only if s lies near some $a_j(u_*)$; thus, the restriction on speed s is only the assumption that the triple (u_-, u_+, s) be associated with the pth and not some other characteristic field.*

PROPOSITION 3.2. *Let Assumptions 3.1–3.5 hold, and let $\hat{u}(x - st)$ be a viscous shock solution such that the profile $\{\hat{u}(z)\}$ lies entirely within a sufficiently small neighborhood $\mathcal{V} \subset \mathcal{U}$ of u_*, and the speed s lies within a sufficiently small neighborhood of $a_p(u_*)$. Then, letting $\epsilon := |u_+ - u_-|$ denote shock strength, and $\delta := \max |u_\pm - u_*|$ the distance from base point u_*, we have bounds*

$$\hat{u}' = \mathcal{O}(\varepsilon^2) e^{-\theta\varepsilon|x|}(r_p(u_*) + \mathcal{O}(\delta)) \tag{3.9}$$

$$\hat{u}'' = \mathcal{O}(\varepsilon^3) e^{-\theta\varepsilon|x|} \tag{3.10}$$

and

$$a_j' = \mathcal{O}(|\hat{u}'|), \tag{3.11}$$

$$a_j'' = \mathcal{O}(|\hat{u}''| + |\hat{u}'|^2) = o(|\hat{u}'|), \tag{3.12}$$

with, moreover,

$$a_p' \leq -\theta|\hat{u}'| \tag{3.13}$$

for some uniform constant $\theta > 0$.

2.2. Linearization. As was done in (2.18), we linearize (3.1) about the profile to get the eigenvalue problem

$$\lambda v = Lv := -[(A+E)v]' + (Bv')', \tag{3.14}$$

where

$$B := B(\hat{u}), \quad A := df(\hat{u}) - sI, \quad \text{and} \quad Ev := -(dBv)\hat{u}_x. \tag{3.15}$$

2.3. Essential Spectrum. According to Theorem 3.1, Assumptions 3.1–3.2 imply that $\Re e\,[\sigma(-i\xi A(u) - |\xi|^2 B(u))] \leq -\theta|\xi|^2/(1+|\xi|^2)$, which means that the essential spectrum of L does not intersect Σ_+.

2.4. Integrated eigenvalue problem. Following Lemma 2.2, we have the integrated eigenvalue problem

$$\lambda W = \mathcal{L}W := -(A+E)W' + BW''. \tag{3.16}$$

3. Stability theorem

THEOREM 3.2. *Let Assumptions 3.1–3.5 hold, and let $\hat{u}(x-st)$ be a viscous shock solution such that the profile $\{\hat{u}(z)\}$ lies entirely within a sufficiently small neighborhood $\mathcal{V} \subset \mathcal{U}$ of u_*, and the speed s lies within a sufficiently small neighborhood of $a_p(u_*)$. Then, $\hat{u}$ is spectrally stable.*

PROOF. In the sections following, we arrive at the two inequalities,

$$\Re e\lambda\|W\|^2 + \|W'\|^2 + \|BW''\|^2 \leq C\int |\hat{u}'||W|^2 \tag{3.17}$$

and

$$\Re e\lambda\|W\|^2 + \int |\hat{u}'||W|^2 \leq C\epsilon\|W'\|^2, \tag{3.18}$$

which hold for sufficiently small ϵ and for $\lambda \in \Sigma_+$. Adding $C\epsilon$ times (3.17) to (3.18) we obtain

$$\Re e\lambda\|W\|^2 + \int |\hat{u}'||W|^2 \leq 0. \tag{3.19}$$

Whence, if $\lambda \in \Sigma_+$, then $W \equiv 0$, which is a contradiction. □

4. Basic energy estimates.

We first derive standard, "Friedrichs-type" estimates for the eigenvalue problem [**7**].

LEMMA 3.1. *Suppose that λ is an eigenvalue of L, $\mathcal{L}$, with $\Re e\lambda \geq 0$, $\lambda \neq 0$. Then, there hold estimates*

$$\Re e\lambda\|W\|^2 + \|BW'\|^2 \leq C\int |\hat{u}'||W|^2, \tag{3.20}$$

$$|\Im m\lambda|\int |\hat{u}'||W|^2 \leq C\int |\hat{u}'|(\eta|W|^2 + \eta^{-1}|W'|^2), \tag{3.21}$$

and

$$\Re e\lambda\|w\|^2 + \|Bw'\|^2 \leq C\int |\hat{u}'||w|^2, \tag{3.22}$$

for some constant $C > 0$, any $\eta > 0$.

PROOF. From (3.15), we have

$$|A'|,\ |E| = \mathcal{O}(|\hat{u}'|). \tag{3.23}$$

Similarly, by Assumption 3.2, the block structure assumption Assumption 3.3, and (3.15), we have

$$v \cdot (A^0 B v) \geq |Bv|^2/C, \tag{3.24}$$

$$|(A^0 B)' v| \leq C|\hat{u}'||Bv|, \tag{3.25}$$

$$|A^0 E v| \leq C|\hat{u}'||Bv|, \tag{3.26}$$

for any vector v, for some constant $C > 0$.

Taking the real part of the L^2 inner product of $A^0 W$ against (3.16), applying (3.15) and (C.1), and integrating the viscous (second-order) term by parts, we thus obtain

$$\begin{aligned}
\Re e\lambda \langle W, A^0 W\rangle &= \Re e\langle W, A^0 B W''\rangle - \Re e\langle W, A^0 E W'\rangle + (1/2)\langle W, (A^0 A)' W\rangle \\
&= -\langle W', A^0 B W'\rangle - \Re e\langle W, [(A^0 B)' - A^0 E] W'\rangle \\
&\quad + (1/2)\langle W, (A^0 A)' W\rangle \\
&= -\langle W', A^0 B W'\rangle + \int \mathcal{O}(|\hat{u}'|)(|BW'|^2 + |W|^2),
\end{aligned}$$

and, rearranging, and absorbing $\mathcal{O}(\int |\hat{u}'||BW'|^2) = \mathcal{O}(\epsilon \|BW'\|^2)$ into the favorable term $-\langle W', A^0 B W'\rangle \leq -\|BW'\|^2/C$, we obtain the claimed inequality (3.20). Inequalities (3.21) and (3.22) follow similarly, with the parameter η arising in (3.21) by an application of Young's inequality. (Note the appearance of multiplier $|\hat{u}'|$ in the lefthand side of (3.21). $\square$

Corollary 3.1. *Suppose that λ is an eigenvalue of L, $\mathcal{L}$, with $\Re e\lambda \geq 0$, $\lambda \neq 0$. Then, $|\Re e\lambda| \leq C\epsilon^2$, for some constant $C > 0$.*

Proof. Otherwise, the right-hand side of (3.20) can be absorbed in the term $\Re e\lambda \|W\|^2$, since $|\hat{u}'| \leq C\epsilon^2$, by (3.9). But, this implies $W \equiv 0$, a contradiction. $\square$

5. Derivative estimate

Next, we carry out a nonstandard derivative estimate of the type formalized by Kawashima [**27**]. The origin of this approach goes back to [**26, 40**] in the context of gas dynamics; see, e.g., [**18**] for further discussion and references.

LEMMA 3.2. *Suppose that λ is an eigenvalue of L, $\mathcal{L}$, with $\Re e\lambda \geq 0$, $\lambda \neq 0$. Then,*

$$\|W'\|^2 \leq C(|\Re e\lambda|\eta\|W\|^2 + \int |\hat{u}'||W|^2 + \|BW''\|^2/\eta), \tag{3.27}$$

for some constant $C > 0$ and $\eta > 0$, ϵ^2/η sufficiently small.

PROOF. Taking the real part of the L^2 inner product of W' against K times (3.16), where K is as in (3.6), applying (C.2) (see appendix C), and using Young's inequality repeatedly, we obtain

$$\begin{aligned}
\Re e(\langle W', KAW'\rangle &= \Re e\big(-\lambda\langle W', KW\rangle - \langle W', KEW'\rangle + \langle W', KBW''\rangle\big) \\
&\leq |\Re e\lambda|\langle |W'|, |KW|\rangle + |\Im m\lambda|\langle |W|, |K'W|\rangle \\
&\quad + \langle |W'|, |KEW'|\rangle + \langle |W'|, |KBW''|\rangle \\
&\leq C\big[|\Re e\lambda|(\|W'\|^2/\eta + \eta\|W\|^2) + |\Im m\lambda| \int |\hat{u}'||W|^2 \\
&\quad + \epsilon^2\|W'\|^2 + (\eta\|W'\|^2 + \|BW''\|/\eta)\big].
\end{aligned}$$

Recalling that $|\Re e\lambda| \leq C\epsilon^2$ by Corollary 3.1, and

$$\|W'\|^2 \leq C\big(\Re e\langle W', KAW'\rangle + \int |\hat{u}'||W|^2\big),$$

by (3.6) combined with (3.20), we find for η, ϵ^2/η sufficiently small that the terms $|\Re e\lambda|\|W'\|^2/\eta$, $C\epsilon\|W'\|^2$, and $C\eta\|W'\|^2$ can up to a term of order $\int |\hat{u}'||W|^2$ be absorbed in the left hand side, yielding

$$\|W'\|^2 \leq C\big(|\Re e\lambda|\eta\|W\|^2|\Im m\lambda| + \int |\hat{u}'||W|^2 + \|BW''\|^2/\eta\big). \tag{3.28}$$

Applying bound (3.21) and recalling that $|\hat u'| \le C\epsilon^2$, we find for η, ϵ^2/η sufficiently small that the term $C|\Im m\lambda|$ on the right-hand side may be absorbed in the lefthand side and $C\int |\hat u'||W|^2$, giving the result. □

Corollary 3.2. *Suppose that λ is an eigenvalue of L, $\mathcal{L}$, with $\Re e\lambda \ge 0$, $\lambda \ne 0$. Then,*

$$\Re e\lambda \|W\|^2 + \|W'\|^2 + \|BW''\|^2 \le C \int |\hat u'||W|^2, \tag{3.29}$$

for some constant $C > 0$, for all ϵ sufficiently small.

Proof. Adding C times (3.20), C/η times (3.22), and (3.27), with $C > 0$ sufficiently large, and η sufficiently small, we obtain the result. (Recall that $BW'' = Bw'$). □

6. Weighted energy estimate.

At this point, we have reduced the problem essentially to the situation of the strictly parabolic case. Evidently, the main issue here, as there, is to control the term $C\int |\hat u'||W|^2$ on the right-hand side of (3.29). This we can accomplish using the weighted energy method of Goodman [**15**] with a bit of extra care.

Corollary 3.3. *Given Assumptions 3.1–3.5, there exist smooth, real matrix-valued functions $\tilde R(u)$, $\tilde L(u)$, $\tilde L\tilde R = I$, such that*

$$\tilde L A \tilde R = \begin{pmatrix} A_- & 0 & 0 \\ 0 & a_p & 0 \\ 0 & 0 & A_+ \end{pmatrix}, \tag{3.30}$$

where $A_- \le a_- < 0$ and $A_+ \ge a_+ > 0$ are symmetric, and

$$\tilde L B \tilde R \ge 0 \tag{3.31}$$

is symmetric, nonnegative definite.

PROOF. Note that Assumption 3.1 implies that $(A^0)^{1/2}A(A^0)^{-1/2}$ is symmetric, and likewise $(A^0)^{1/2}B(A^0)^{-1/2}$ is symmetric, nonnegative definite. By Assumption 3.4, there is spectral separation between eigenvalue a_p and the positive and negative spectra of matrix $(A^0)^{1/2}A(A^0)^{-1/2}$, hence it can be block diagonalized by a real, orthogonal transformation $O(A^0)^{1/2}A(A^0)^{-1/2}O^t$, $O^t = O^{-1}$, which likewise preserves symmetry, and semidefinite positivity of $(A^0)^{1/2}B(A^0)^{-1/2}$. Setting $\tilde{R} = (A^0)^{-1/2}O^t$, $\tilde{L} = O(A^0)^{1/2}$, we are done. □

LEMMA 3.3. *Given Assumptions 3.1–3.5, there exist smooth, real matrix-valued functions $R(u)$, $L(u)$, $LR = I$, such that*

$$LAR = \begin{pmatrix} A_- & 0 & 0 \\ 0 & a_p & 0 \\ 0 & 0 & A_+ \end{pmatrix}, \tag{3.32}$$

where $A_- \leq a_- < 0$ and $A_+ \geq a_+ > 0$ are symmetric,

$$(LR')_{pp} = (L'R)_{pp} = 0, \tag{3.33}$$

and

$$\Re e LBR \geq -C\epsilon \tag{3.34}$$

for some constant $C > 0$.

PROOF. Set $R := \Gamma\tilde{R}$, $L := \Gamma^{-1}\tilde{L}$, with

$$\Gamma := \begin{pmatrix} I_{p-1} & 0 & 0 \\ 0 & \gamma & 0 \\ 0 & 0 & I_{n-p} \end{pmatrix}, \tag{3.35}$$

and define γ by the linear ODE

$$\gamma' = -\tilde{l}_p\tilde{r}_p'\gamma; \quad \gamma(0) = 1, \tag{3.36}$$

where $\tilde{l}_p$, $\tilde{r}_p$ denote the pth row and column, respectively, of $\tilde{L}$, $\tilde{R}$. Clearly, L and R still block-diagonalize A in the manner claimed, while

$$(LR')_{pp} = \gamma^{-1}\tilde{l}_p(\gamma\tilde{r}_p)' = \gamma^{-1}\tilde{l}_p(\gamma'\tilde{r}_p + \gamma\tilde{r}_p')$$
$$= \gamma^{-1}(\gamma' + \gamma(\tilde{l}_p\tilde{r}_p')) = 0,$$

by (3.36). On the other hand, $|r_p'| \le C|\hat{u}'|$, whence we obtain by direct integration of (3.36) the bound

$$\gamma(x) = e^{\int_0^x -\ell_p r_p'} = e^{\mathcal{O}(\int_{-\infty}^{+\infty}|\hat{u}'|)}$$
$$= e^{\mathcal{O}(\epsilon)} = 1 + \mathcal{O}(\epsilon),$$

yielding bound (3.34) by (3.31) and continuity. □

Lemma 3.4. *Let there hold Assumptions 3.1–3.5, and suppose that λ is an eigenvalue of L, $\mathcal{L}$, with $\Re e\lambda \ge 0$, $\lambda \ne 0$, and the shock strength ϵ sufficiently small. Then,*

$$\Re e\lambda\|W\|^2 + \int |\hat{u}'||W|^2 \le C\epsilon\|W'\|^2, \tag{3.37}$$

for some constant $C > 0$.

Proof. By the construction described above, we have, clearly:

$$|L'|, |R'| = \mathcal{O}(\hat{u}'), \tag{3.38}$$

$$|L''|, |R''| = \mathcal{O}(|\hat{u}''| + |\hat{u}'|^2). \tag{3.39}$$

Setting $Z := LW$, and left multiplying (3.16) by L, we thus obtain

$$\lambda Z + (\bar{A} + \bar{E})Z' + \bar{M}Z = (\bar{B}Z')' \tag{3.40}$$

where $\bar{A} := LAR$ is as in (3.32), $\bar{B} := LBR > -C\epsilon$, $\bar{E}$ defined by

$$\bar{E}v := LBR'v - L'BRv + L(dB\hat{u}_x)Rv - L(dBRv)\hat{u}_x \tag{3.41}$$

satisfies

$$\bar{E} = \mathcal{O}(|\hat{u}'|), \quad \bar{E}' = \mathcal{O}(|\hat{u}''| + |\hat{u}'|^2) = \mathcal{O}(\epsilon|\hat{u}'|), \tag{3.42}$$

and $\bar{M}$ defined by

$$\bar{M}v := \bar{A}LR'v + L(dB\hat{u}_x)R'v - L(dBR'v)\hat{u}_x - L(BR')'v \tag{3.43}$$

satisfies

$$|\bar{M}| = \mathcal{O}(\hat{u}'), \quad |\bar{M}_{pp}| = \mathcal{O}(|\hat{u}''| + |\hat{u}'|^2) = \mathcal{O}(\epsilon|\hat{u}'|), \tag{3.44}$$

the second estimate following by normalization (3.33). Clearly, to establish (3.37), it is sufficient to establish the corresponding result in Z coordinates.

Following [**15**], define weight $\alpha_p \equiv 1$, and define weights $\alpha_\pm$ by ODE

$$\alpha_\pm' = -C|\hat{u}'|\alpha_\pm/a_\pm, \quad \alpha_\pm(0) := 1, \tag{3.45}$$

whence

$$\begin{aligned} \alpha_\pm(x) = e^{\int_0^x C|\hat{u}'|/a_\pm} &= 1 + \mathcal{O}(C\int_{-\infty}^{\infty}|\hat{u}'|) \\ &= 1 + \mathcal{O}(C\epsilon) = \mathcal{O}(1), \end{aligned}$$

$$\alpha_j' = \mathcal{O}(|\hat{u}'|), \quad j = -, p, +. \tag{3.46}$$

Here, C is a sufficiently large constant to be chosen later, and ϵ is so small that $\mathcal{O}(C\epsilon) < 1$. Set $\alpha := \operatorname{diag}\{\alpha_j\}$.

Now, take the real part of the complex L^2 inner product of αZ with (3.40), to obtain the energy estimate (after integration by parts)

$$\begin{aligned} \Re e\lambda \sum \int \alpha_j |Z_j|^2 - \sum \langle Z_j, (a_j\alpha_j)' Z_j\rangle + \Re e \int \langle Z', \alpha \bar{B} Z'\rangle = \\ \Re e \int \langle Z, \alpha \bar{M} Z\rangle - \Re e \int \langle \alpha Z, \bar{E} Z'\rangle - \Re e \int \langle \alpha' Z, \bar{B} Z'\rangle, \end{aligned}$$

where j is summed over $-, p, +$, and $Z =: (Z_-, Z_p, Z_+)^t$.

Noting that

$$\begin{cases} (\alpha_p a_p)' = a_p' < -\theta|\hat{u}'|, \\ (\alpha_j a_j)' = \alpha_j' a_j + \alpha_j a_j' \\ \qquad\qquad < -C\theta|\hat{u}'| \; for \; j \neq p, \end{cases} \tag{3.47}$$

where C may be chosen arbitrarily large, and that $\Re e \alpha \bar{B} > -C\epsilon$ by continuity, for ϵ sufficiently small, and using estimates (3.42)–(3.43) to absorb all terms in the right-hand side, we obtain the result. More precisely, we have used Young's inequality to bound the second and third terms on the right-hand side of (3.47) by

$$C \int |\hat{u}'||Z||Z'| \leq \frac{C}{2}\left(\int |\hat{u}'|^{3/2}|Z|^2 + \int |\hat{u}'|^{1/2}|Z'|^2\right) \tag{3.48}$$

$$\leq \frac{C}{2}\left(\epsilon \int |\hat{u}'||Z|^2 + \epsilon \int |Z'|^2\right), \tag{3.49}$$

a contribution that is clearly absorbable on the left-hand side. The first term on the right-hand side is bounded by

$$C_2\left(\epsilon \int |\hat{u}'||Z_p|^2 + \int |\hat{u}'||Z_\pm|^2\right), \tag{3.50}$$

where C_2 is some fixed constant, hence it is also absorbable. □

CHAPTER 4

Stability of relaxation shocks

1. Introduction

In this chapter, we consider the Jin-Xin relaxation model [**24**]:

$$\begin{aligned} U_t + V_x &= 0, \\ V_t + AU_x &= f(U) - V, \end{aligned} \tag{4.1}$$

where $f, U, V \in \mathbb{R}^n$, $f \in C^3$, $x \in \mathbb{R}$, and $A \in \mathbb{R}^{n\times n}$ is constant. This system falls into the general class of relaxation equations,

$$\begin{aligned} U_t + F(U,V)_x &= 0, \\ V_t + G(U,V)_x &= Q(U,V), \end{aligned} \tag{4.2}$$

which has roots in kinetic theory. In addition (4.1) serves as the basis for an important numerical scheme for approximating solutions of hyperbolic conservation laws

$$U_t + f(U)_x = 0.$$

In Chapter 2 we showed that traveling wave solutions of (4.1) correspond to those for the viscous conservation laws, see Example 2.3. Our aim in this section is to examine the stability problem for the Jin-Xin relaxation model and compare our results to those of the previous chapter.

Let $\mathcal{U}$ be a neighborhood of a particular base point (u_*, v_*). We assume the following:

Assumption 4.1 (Symmetrizability). *For all $(u,v) \in \mathcal{U}$, there exists a symmetrizer $A^0(u,v)$, symmetric and positive definite, such that $A^0\left(\begin{smallmatrix} F_u & F_v \\ G_u & G_v \end{smallmatrix}\right)$ is symmetric and $A^0\left(\begin{smallmatrix} 0 & 0 \\ Q_u & Q_v \end{smallmatrix}\right)$ is both symmetric and nonpositive definite.*

Let

$$a_1(u) \leq \cdots \leq a_n(u) \tag{4.3}$$

denote the (real) eigenvalues of $df(u,v)$. Let $R(u,v) = [r_j(u,v)]$ and $L(u,v) = [l_j(u,v)]$ be a smooth choice of associated right and left eigenvectors, respectively, satisfying $l_j \cdot r_k = \delta_{jk}$.

Assumption 4.2 (Subcharacteristic condition). *For $(u,v) \in \mathcal{U}$, we have*

$$\tilde{A} - \Lambda^2 > 0, \tag{4.4}$$

where $\tilde{A} := LAR$ and $\Lambda := Ldf R$ (note that Λ is diagonal).

Remark 4.1. *Note that these two assumptions are similar to those of the previous chapter. Indeed, the general form of the Kawashima class, which is given in Appendix A, allows for certain relaxation models. We note also that Assumption 4.2 implies genuine coupling and that the block structure requirement is automatically satisfied by the Jin-Xin system.*

We further assume:

Assumption 4.3 (Simplicity). *The pth characteristic field is of multiplicity one, i.e. $a_p(u_*, v_*)$ is a simple eigenvalue of $df(u_*, v_*)$.*

Assumption 4.4 (Genuine nonlinearity). *The pth characteristic field is genuinely nonlinear, i.e. $\nabla a_p \cdot r_p(u_*, v_*) \neq 0$.*

Remark 4.2. *In appendix B, we prove the following: Given Assumption 4.2, we have that Assumption 4.1 is equivalent to the statement that A and df can be simultaneously diagonalized, i.e., that*

$$[A, df] = 0.$$

Hence, without loss of generality, we have that (4.4) is diagonal.

In this chapter, we show that small-amplitude shocks are spectrally stable. Hence, combined with the recent work of Mascia and Zumbrun [**38**], we conclude that small-amplitude shocks are orbitally stable.

In Jin and Xin's original work [**24**], they showed this system to have an L^1 contraction property for scalar shocks ($n = 1$), which implies orbital stability. By using energy methods, we also show that the scalar case is spectrally stable. However, a generalization of our method to a "Goodman-type" weighted norm estimate extends our scalar result to higher dimensions for small-amplitude shocks. H. Liu [**33**] recently proved orbital stability under zero-mass perturbations, a result slightly more general than this one. However, in light of Mascia and Zumbrun's recent work, one can get from spectral stability to the more general orbital stability directly, and so much of Liu's analysis can be avoided.

Recently, Godillon [**13**] carried out stability index calculations for the Jin-Xin model, which are consistent with stability for small-amplitude shocks. While this is an encouraging result, consistency only serves as a necessary condition for stability. Since our stability results only hold generally in the small-amplitude limit, other techniques will need to be explored to expand these results to larger amplitude shocks, e.g., numerical Evans function calculations [**2**].

2. Preliminaries

2.1. Existence and asymptotics. Since the profiles for (4.1) and (3.1) are the same, we can apply the results given in Propositions 3.1 and 3.2 to the Jin-Xin shocks. As a result we have the following asymptotic bounds:

$$\hat{u}' = \mathcal{O}(\varepsilon^2)e^{-\theta\varepsilon|x|}(r_p(u_*, v_*) + \mathcal{O}(\delta)) \tag{4.5}$$

$$\hat{u}'' = \mathcal{O}(\varepsilon^3)e^{-\theta\varepsilon|x|} \tag{4.6}$$

and

$$a_j' = \mathcal{O}(|\hat{u}'|), \tag{4.7}$$

$$a_j'' = \mathcal{O}(|\hat{u}''| + |\hat{u}'|^2) = o(|\hat{u}'|), \tag{4.8}$$

with, moreover,

$$a_p' \le -\theta|\hat{u}'| \tag{4.9}$$

for some uniform constant $\theta > 0$.

2.2. Linearization. As was done in (2.18), we linearize (4.1) about the profile to get the eigenvalue problem

$$\begin{aligned} \lambda U - sU' + V' &= 0, \\ \lambda V - sV' + AU' &= Df(\hat{U})U - V. \end{aligned} \tag{4.10}$$

2.3. Essential Spectrum. According to the work of Kawashima, see Appendix A, Assumptions 4.1–4.2 imply that

$$\Re e\,[\sigma(-i\xi dF(u,v) + dQ(u,v))] \le -\theta|\xi|^2/(1+|\xi|^2),$$

which means that the essential spectrum of L does not intersect Σ_+.

2.4. Integrated eigenvalue problem. For relaxation, we can not immediately use Lemma 2.2. Instead, we integrate in U only. Hence, we have the Jin-Xin integrated eigenvalue problem

$$\begin{aligned} \lambda U - sU' + V &= 0, \\ \lambda V - sV' + AU'' &= Df(\hat{U})U' - V. \end{aligned} \tag{4.11}$$

3. Spectral stability of scalar case

In this section we prove that the scalar, $n = 1$, eigenvalue equation (4.10) exhibits spectral stability. The integrated coordinate eigenvalue problem (4.11) takes the form

$$\lambda u - su' + v = 0, \tag{4.12a}$$

$$\lambda v - sv' + Au'' = f'(\hat{u})u' - v, \tag{4.12b}$$

where $u, v \in \mathbf{R}$, $\hat{u}$ is the profile, $\hat{u}_x < 0$, $f'' > 0$, and $A > f'(\hat{u})^2$.

THEOREM 4.1. *Scalar Jin-Xin shocks exhibit spectral stability.*

PROOF. We refer to Lemma 4.1 below for the following identities, which hold for any point spectra satisfying $\Re e\lambda \geq 0$:

$$(i) \quad \int_{-\infty}^{+\infty} |v|^2 \leq \int_{-\infty}^{+\infty} |f'(\hat{u})u'\bar{v}|,$$

$$(ii) \quad \int_{-\infty}^{+\infty} A|u'|^2 < \int_{-\infty}^{+\infty} |v|^2.$$

By adding half of (ii) to (i), we get

$$\frac{1}{2}\int_{-\infty}^{+\infty} (|v|^2 + A|u'|^2) < \int_{-\infty}^{+\infty} |f'(\hat{u})u'\bar{v}|,$$

which by Young's inequality yields

$$\frac{1}{2}\int_{-\infty}^{+\infty} (|v|^2 + A|u'|^2) < \frac{1}{2}\int_{-\infty}^{+\infty} (|v|^2 + f'(\hat{u})^2|u'|^2).$$

Since $A \geq f'(\hat{u})^2$, see (4.4), this is a contradiction. $\square$

LEMMA 4.1. *For $\Re e\lambda \geq 0$, (i) and (ii) in the above proof hold.*

PROOF. (i) We begin by multiplying (4.12b) by the conjugate $\bar{v}$ and integrating from $-\infty$ to ∞. We get

$$(\lambda + 1)\int_{-\infty}^{+\infty} |v|^2 - s\int_{-\infty}^{+\infty} v'\bar{v} + \int_{-\infty}^{+\infty} Au''\bar{v} = \int_{-\infty}^{+\infty} f'(\hat{u})u'\bar{v}.$$

We take the real part and note that the second term vanishes, leaving us with

$$(\Re e(\lambda) + 1)\int_{-\infty}^{+\infty} |v|^2 + \Re e(\int_{-\infty}^{+\infty} Au''\bar{v}) = \Re e(\int_{-\infty}^{+\infty} f'(\hat{u})u'\bar{v}).$$

Finally, by replacing $\bar{v}$ with $s\bar{u}' - \bar{\lambda}\bar{u}$ from (4.12a) and appropriately integrating by parts, we arrive at

$$\Re e(\lambda)\int_{-\infty}^{+\infty}(|v|^2 + A|u'|^2) + \int_{-\infty}^{+\infty}|v|^2 = \Re e(\int_{-\infty}^{+\infty} f'(\hat{u})u'\bar{v}).$$

Thus, for $\Re e\lambda \geq 0$, we have

$$\int_{-\infty}^{+\infty}|v|^2 \leq \Re e(\int_{-\infty}^{+\infty} f'(\hat{u})u'\bar{v}) \leq \int_{-\infty}^{+\infty}|f'(\hat{u})u'\bar{v}|.$$

(ii) We construct this identity by multiplying (4.12b) by the conjugate $\bar{u}$ and integrating from $-\infty$ to ∞. We get

$$(\lambda+1)\int_{-\infty}^{+\infty} v\bar{u} - s\int_{-\infty}^{+\infty} v'\bar{u} + \int_{-\infty}^{+\infty} Au''\bar{u} = \int_{-\infty}^{+\infty} f'(\hat{u})u'\bar{u}.$$

Integrating the second and third terms by parts and adjusting terms yields

$$(\lambda+\bar{\lambda}+1)\int_{-\infty}^{+\infty} v\bar{u} - \int_{-\infty}^{+\infty} v(\bar{\lambda}\bar{u} - s\bar{u}') = \int_{-\infty}^{+\infty} A|u'|^2 + \int_{-\infty}^{+\infty} f'(\hat{u})u'\bar{u},$$

which gives

$$(2\Re e(\lambda)+1)\int_{-\infty}^{+\infty} v\bar{u} + \int_{-\infty}^{+\infty}|v|^2 = \int_{-\infty}^{+\infty} A|u'|^2 + \int_{-\infty}^{+\infty} f'(\hat{u})u'\bar{u}.$$

Now, take the real part:

$$\begin{aligned}(2\Re e(\lambda)+1)\Re e(\int_{-\infty}^{+\infty} v\bar{u}) &+ \int_{-\infty}^{+\infty}|v|^2 \\ &= \int_{-\infty}^{+\infty} A|u'|^2 - \frac{1}{2}\int_{-\infty}^{+\infty} f''(\hat{u})\hat{u}_x|u|^2.\end{aligned} \tag{4.13}$$

By using (4.12a), we observe that

$$\lambda\int_{-\infty}^{+\infty}|u|^2 - s\int_{-\infty}^{+\infty} u'\bar{u} + \int_{-\infty}^{+\infty} v\bar{u} = 0.$$

Hence, by taking the real part, we have

$$\Re e(\int_{-\infty}^{+\infty} v\bar{u}) = -\Re e(\lambda)\int_{-\infty}^{+\infty}|u|^2,$$

which goes into (4.13) to give

$$\int_{-\infty}^{+\infty} |v|^2 = \Re e(\lambda)((2\Re e(\lambda)+1)) \int_{-\infty}^{+\infty} |u|^2 + \int_{-\infty}^{+\infty} A|u'|^2 - \frac{1}{2}\int_{-\infty}^{+\infty} f''(\hat{u})\hat{u}_x|u|^2.$$

Thus for $\Re e\lambda \geq 0$, we have

$$\int_{-\infty}^{+\infty} A|u'|^2 < \int_{-\infty}^{+\infty} |v|^2.$$

□

4. Spectral stability of systems

In this section we show that the eigenvalue problem (4.10) exhibits spectral stability for small-amplitude shocks. From (4.11), we have

$$\begin{aligned} &\lambda U - sU' + V = 0, \\ &\lambda V - sV' + AU'' = Df(\hat{U})U' - V. \end{aligned} \tag{4.14}$$

Recall from (4.2) and (4.4) that L and R diagonalize $Df(u_*, v_*)$, and thus there exist a C^2 choice of R,L in a neighborhood of the base point (u_*, v_*) satisfying $\tilde{A} = LAR > \Lambda^2$. By transforming $U \to RU$ and $V \to RV$, we have

$$\lambda RU - s(RU' + R'U) + RV = 0,$$

$$\lambda RV - s(RV' + R'V) + R\tilde{A}L(R''U + 2R'U' + RU'') = R\Lambda L(R'U + RU') - RV.$$

Left multiplying by L yields

$$\lambda U - s(U' + LR'U) + V = 0, \tag{4.15a}$$

$$\lambda V - s(V' + LR'V) + \tilde{A}(LR''U + 2LR'U' + U'') = \Lambda(LR'U + U') - V. \tag{4.15b}$$

Following the analysis of Goodman [**14**] (see Lemma 3.3, also [**52**], [**38**], [**23**]), we can scale L and R so that in addition,

$$(LR')_{pp} = 0. \tag{4.16}$$

THEOREM 4.2. *Small-amplitude Jin-Xin shocks exhibit spectral stability.*

PROOF. We refer to Lemma 4.2 below for the following inequalities, which hold in the small-amplitude shock limit for $\Re e\lambda \geq 0$, where u_j, v_j denote coordinates of (U,V) in (4.15) and L, R chosen as in (4.2), with the rescaling of (4.16):

$$\begin{aligned}
(i) \quad & \sum_{j=1}^{n}\int_{-\infty}^{+\infty} \alpha_j |v_j|^2 \leq \sum_{j=1}^{n}\int_{-\infty}^{+\infty} |\alpha_j \Lambda_j||v_j||u_j'| \\
& \quad + C_1 \int_{-\infty}^{+\infty} |\hat{U}_x| \left[\epsilon_1 |u_p|^2 + \sum_{j\neq p} |u_j|^2 + \sum_{j=1}^{n}(|v_j|^2 + |u_j'|^2)\right],
\end{aligned}$$

$$\begin{aligned}
(ii) \quad & \sum_{j=1}^{n}\int_{-\infty}^{+\infty} \alpha_j \tilde{A}_j |u_j'|^2 + \frac{1}{2}\sum_{j=1}^{n}\int_{-\infty}^{+\infty} (s\alpha_j' - (\alpha_j\Lambda_j)')|u_j|^2 \\
& \leq \sum_{j=1}^{n}\int_{-\infty}^{+\infty} \alpha_j |v_j|^2 + C_2 \int_{-\infty}^{+\infty} |\hat{U}_x| \left[\epsilon_2 |u_p|^2 + \frac{1}{\epsilon_2}\sum_{j\neq p} |u_j|^2 + \sum_{j=1}^{n} |u_j'|^2\right],
\end{aligned}$$

where $\alpha = \mathrm{diag}(\alpha_1, \alpha_2, ..., \alpha_n)$, is a positive-diagonal matrix satisfying $\alpha_p = 1$ and for $j \neq p$,

$$\begin{aligned}
\alpha_j'(x) &= \frac{-C_3}{\Lambda_j - s}|\hat{U}_x|\alpha_j(x), \\
\alpha_j(0) &= 1.
\end{aligned}$$

Just as with the scalar case, we add half of (ii) to (i) and simplify to get

$$\begin{aligned}
& \frac{1}{2}\sum_{j=1}^{n}\int_{-\infty}^{+\infty} \alpha_j(|v_j|^2 + \tilde{A}_j|u_j|^2) + \frac{1}{2}\sum_{j=1}^{n}\int_{-\infty}^{+\infty} (s\alpha_j' - (\alpha_j\Lambda_j)')|u_j|^2 \\
& \leq C_4 \int_{-\infty}^{+\infty} |\hat{U}_x| \left[\epsilon_3 |u_p|^2 + \frac{1}{\epsilon_3}\sum_{j\neq p} |u_j|^2 + \sum_{j=1}^{n}(|v_j|^2 + |u_j'|^2)\right] \\
& \quad + \sum_{j=1}^{n}\int_{-\infty}^{+\infty} |\alpha_j\Lambda_j||v_j||u_j'|.
\end{aligned} \tag{4.17}$$

We claim that for C_3 sufficiently large and ϵ_3, $|\hat{U}_x|$ sufficiently small,

$$\frac{1}{2}\sum_{j=1}^{n}\int_{-\infty}^{+\infty}(s\alpha_j' - (\alpha_j\Lambda_j)')|u_j|^2 \geq C_4\int_{-\infty}^{+\infty}|\hat{U}_x|\left[\epsilon_3|u_p|^2 + \frac{1}{\epsilon_3}\sum_{j\neq p}|u_j|^2\right].$$

For $j = p$ and sufficiently small ϵ_3, we have

$$-\frac{1}{2}\int_{-\infty}^{+\infty}\Lambda_p'|u_p|^2 \geq \epsilon_3 C_4\int_{-\infty}^{+\infty}|\hat{U}_x||u_p|^2,$$

since $\Lambda_p' \geq -\theta|\hat{U}_x|$. For $j \neq p$ and C_3 sufficiently large,

$$\begin{aligned}\frac{1}{2}\int_{-\infty}^{+\infty}(s\alpha_j' - (\alpha_j\Lambda_j)')|u_j|^2 &= \frac{1}{2}\int_{-\infty}^{+\infty}((s-\Lambda_j)\alpha_j' - \alpha_j\Lambda_j'))|u_j|^2,\\ &= \frac{1}{2}\int_{-\infty}^{+\infty}(C_3|\hat{U}_x|\alpha_j - \alpha_j\Lambda_j'))|u_j|^2,\\ &\geq \frac{C_4}{\epsilon_3}\int_{-\infty}^{+\infty}|\hat{U}_x||u_j|^2.\end{aligned}$$

Thus, (4.17) becomes

$$\begin{aligned}\frac{1}{2}\sum_{j=1}^{n}\int_{-\infty}^{+\infty}\alpha_j(|v_j|^2 + \tilde{A}_j|u_j'|^2) &\leq C_4\sum_{j=1}^{n}\int_{-\infty}^{+\infty}|\hat{U}_x|(|v_j|^2 + |u_j'|^2)\\ &+ \sum_{j=1}^{n}\int_{-\infty}^{+\infty}|\alpha_j\Lambda_j||v_j||u_j'|.\end{aligned}$$

Now since $\tilde{A} - \Lambda^2 > 0$, $\exists\eta > 0$ such that $\tilde{A} - (1+\eta)\Lambda^2 > \eta I$. Thus, by Young's inequality,

$$\begin{aligned}\frac{1}{2}\sum_{j=1}^{n}\int_{-\infty}^{+\infty}\alpha_j(|v_j|^2 + \tilde{A}_j|u_j'|^2) &\leq C_4\sum_{j=1}^{n}\int_{-\infty}^{+\infty}|\hat{U}_x|(|v_j|^2 + |u_j'|^2)\\ &+ \frac{1}{2}\sum_{j=1}^{n}\int_{-\infty}^{+\infty}\alpha_j\left[\frac{1}{1+\eta}|v_j|^2 + (1+\eta)|\Lambda_j|^2|u_j'|^2\right],\end{aligned}$$

which simplifies to

$$\frac{1}{2}\sum_{j=1}^{n}\int_{-\infty}^{+\infty}\alpha_j\left[\frac{\eta}{1+\eta}|v_j|^2 + \eta|u_j'|^2\right] \leq C_4\sum_{j=1}^{n}\int_{-\infty}^{+\infty}|\hat{U}_x|(|v_j|^2 + |u_j'|^2).$$

However, since $\alpha_j = 1 + \mathcal{O}(\epsilon)$, then in the small-amplitude shock limit,

$$\frac{\eta}{1+\eta}\alpha_j >> 2C_4\int_{-\infty}^{+\infty}|\hat{U}_x| = \mathcal{O}(\epsilon),$$

$\forall j$. This is a contradiction. Thus $\Re e\lambda < 0$. $\square$

LEMMA 4.2. *For $\Re e\lambda \geq 0$, and L,R chosen as in* (4.16), (*i*) *and* (*ii*) *in the above proof hold, for small-amplitude shocks.*

PROOF. (*i*) We begin by taking the L^2 inner product of (4.15b) with αV to get

$$\begin{aligned}&\langle \alpha V, (\lambda+1)V - s(V' + LR'V) + \tilde{A}(LR''U + 2LR'U' + U'')\rangle\\ &\quad = \langle \alpha V, \Lambda(U' + LR'U)\rangle.\end{aligned}$$

This simplifies to

$$\begin{aligned}&(\lambda+1)\langle \alpha V, V\rangle - s\langle \alpha V, LR'V\rangle - s\langle \alpha V, V'\rangle\\ &\quad = \langle \alpha V, (\Lambda LR' - \tilde{A}LR'')U\rangle + \langle \alpha V, (\Lambda - 2\tilde{A}LR')U'\rangle - \langle \alpha V, \tilde{A}U''\rangle.\end{aligned}$$

Integrating the last term by parts and simplifying gives

$$\begin{aligned}&(\lambda+1)\langle V, \alpha V\rangle - s\langle V, \alpha LR'V\rangle - s\langle V, \alpha V'\rangle\\ &\quad = \langle V, \alpha(\Lambda LR' - \tilde{A}LR'')U\rangle + \langle V, (\alpha\Lambda - 2\alpha\tilde{A}LR' + (\alpha\tilde{A})')U'\rangle\\ &\qquad + \langle V', \alpha\tilde{A}U')\rangle.\end{aligned}$$

By writing V' in terms of U and its derivatives from (4.15a), we have

$$\begin{aligned}&(\lambda+1)\langle V, \alpha V\rangle - s\langle V, \alpha LR'V\rangle - s\langle V, \alpha V'\rangle\\ &\quad = \langle V, \alpha(\Lambda LR' - \tilde{A}LR'')U\rangle + \langle V, (\alpha\Lambda - 2\alpha\tilde{A}LR' + (\alpha\tilde{A})')U'\rangle\\ &\qquad + \langle s((LR')'U + LR'U' + U'') - \lambda U', \alpha\tilde{A}U'\rangle.\end{aligned}$$

Take the real part:

$$\begin{aligned}&\sum_{j=1}^{n}\int_{-\infty}^{+\infty}\left[(\Re e(\lambda)+1)\alpha_j|v_j|^2 + \Re e(\lambda)\alpha_j A_j|u_j'|^2\right]\\ &\quad = s\Re e\langle V, \alpha LR'V\rangle + \Re e\langle V, \alpha(\Lambda LR' - \tilde{A}LR'')U\rangle\\ &\qquad + \Re e\langle V, \alpha\Lambda U\rangle - \Re e\langle V, (2\alpha\tilde{A}LR' - (\alpha\tilde{A})')U'\rangle\\ &\qquad + s\Re e\langle (LR')'U + LR'U', \alpha\tilde{A}U'\rangle - s\sum_{j=1}^{n}\int_{-\infty}^{+\infty}\alpha'(|v_j|^2 + |u_j'|^2).\end{aligned}$$

Note that in the small shock limit, $a', LR' = \mathcal{O}(|\hat{U}_x|)$ and $LR'' = \mathcal{O}(|\hat{U}_{xx}| + |\hat{U}_x|^2)$. Thus for $\Re e\lambda \geq 0$ we have

$$\sum_{j=1}^{n} \int_{-\infty}^{+\infty} \alpha_j |v_j|^2 + \leq \sum_{j=1}^{n} \int_{-\infty}^{+\infty} |\alpha_j \Lambda_j||v_j||u_j'|$$
$$+ \sum_{j=1}^{n} \int_{-\infty}^{+\infty} \mathcal{O}(|\hat{U}_x|)(|v_j|^2 + |u_j'|^2) + \sum_{i,j} \int_{-\infty}^{+\infty} \mathcal{O}(|\hat{U}_x|)|v_i||u_j|.$$

Finally, by Young's inequality, we get

$$\sum_{j=1}^{n} \int_{-\infty}^{+\infty} \alpha_j |v_j|^2 \leq \sum_{j=1}^{n} \int_{-\infty}^{+\infty} |\alpha_j \Lambda_j||v_j||u_j'|$$
$$+ C_1 \int_{-\infty}^{+\infty} |\hat{U}_x| \left[\epsilon_1 |u_p|^2 + \sum_{j \neq p} |u_j|^2 + \sum_{j=1}^{n} (|v_j|^2 + |u_j'|^2) \right].$$

(ii) Now take the L^2 inner product of (4.15b) with αU. We get

$$\langle \alpha U, (\lambda + 1)V - s(V' + LR'V) + \tilde{A}(LR''U + 2LR'U' + U'') \rangle$$
$$= \langle \alpha U, \Lambda(LR'U + U') \rangle.$$

Simplifying yields

$$\langle \alpha U, (\lambda + 1)V - sLR'V \rangle + s\langle \alpha' U, V \rangle + \langle sU', \alpha V \rangle$$
$$= \langle \alpha U, (\Lambda LR' - \tilde{A}LR'')U \rangle + \langle \alpha U, (\Lambda - 2\tilde{A}LR')U' \rangle - \langle \alpha U, \tilde{A}U'' \rangle.$$

Integrating the last term by parts and simplifying gives

$$\langle U, (\lambda + \bar{\lambda} + 1)\alpha - s\alpha LR' + s\alpha')V \rangle + \langle sU' - \lambda U, \alpha V \rangle$$
$$= \langle U, (\alpha \Lambda LR' - \alpha \tilde{A}LR'')U \rangle + \langle U, (\alpha \Lambda - 2\alpha \tilde{A}LR' + (\alpha \tilde{A})')U' \rangle$$
$$+ \langle U', \alpha \tilde{A}U' \rangle.$$

By writing V in terms of U and its derivatives from (4.15a), we have

$$\begin{aligned}&\langle U, ((2\Re e(\lambda)+1)\alpha - s(\alpha LR')^* - s\alpha LR' + s\alpha')(s(U' + LR'U) - \lambda U)\rangle\\ &\quad + \langle V, \alpha V\rangle = \langle U, (\alpha\Lambda LR' - \alpha\tilde{A}LR'')U\rangle\\ &\quad\; + \langle U, (\alpha\Lambda - 2\alpha\tilde{A}LR' + (\alpha\tilde{A})')U'\rangle + \langle U', \alpha\tilde{A}U'\rangle.\end{aligned}$$

Let

$$\begin{aligned}&E = -s\alpha LR' - s(\alpha LR')^* + s\alpha',\\ &N = sE + 2\alpha\tilde{A}LR' - (\alpha\tilde{A})',\\ &M = (\alpha\Lambda LR' - \alpha\tilde{A}LR'') + \lambda E - ((2\Re e(\lambda)+1)\alpha + E)sLR'.\end{aligned}$$

Then we have

$$\begin{aligned}&\langle U, ((2\Re e(\lambda)+1)s\alpha - \alpha\Lambda)U'\rangle + \langle U, NU'\rangle + \langle V, \alpha V\rangle\\ &\quad = \lambda\langle U, (2\Re e(\lambda)+1)\alpha U\rangle + \langle U', \alpha\tilde{A}U'\rangle + \langle U, MU\rangle.\end{aligned}$$

Take the real part:

$$\begin{aligned}&-\frac{1}{2}\langle U, ((2\Re e(\lambda)+1)s\alpha' - (\alpha\Lambda)')U\rangle + \Re e\langle U, NU'\rangle + \langle V, \alpha V\rangle\\ &\quad = \Re e(\lambda)(2\Re e(\lambda)+1)\langle U, \alpha U\rangle + \langle U', \alpha\tilde{A}U'\rangle + \Re e\langle U, MU\rangle.\end{aligned}$$

In the small-amplitude shock limit, N and M are $\mathcal{O}(|\hat{U}_x|)$, while N' is $\mathcal{O}(|\hat{U}_{xx}|+|\hat{U}_x|^2)$. Thus by Young's inequality, all the $|u_p|^2$ terms in N can be made arbitrarily small. The $|(LR')_{pp}||u_1|^2$ terms vanish by (4.16). Thus, all the terms can be absorbed to give

$$\begin{aligned}&\sum_{j=1}^{n}\int_{-\infty}^{+\infty}\alpha_j\tilde{A}_j|u_j'|^2 + \frac{1}{2}\sum_{j=1}^{n}\int_{-\infty}^{+\infty}(s\alpha_j' - (\alpha_j\Lambda_j)')|u_j|^2\\ &\quad \le \sum_{j=1}^{n}\int_{-\infty}^{+\infty}\alpha_j|v_j|^2 + C_2\int_{-\infty}^{+\infty}|\hat{U}_x|\left[\epsilon_2|u_p|^2 + \frac{1}{\epsilon_2}\sum_{j\neq p}|u_j|^2 + \sum_{j=1}^{n}|u_j'|^2\right].\end{aligned}$$

$\square$

CHAPTER 5

Stability of viscous dispersive shocks

1. Introduction

In this chapter, we examine the shock wave spectrum for the following class of systems from isentropic gas dynamics:

$$
\begin{aligned}
&v_t - u_x = 0, \\
&u_t + p(v)_x = (b(v)u_x)_x + dv_{xxx},
\end{aligned}
\tag{5.1}
$$

where physically, v is the specific volume, u is the velocity in Lagrangian coordinates, $p(v)$ is the pressure law for an ideal gas, i.e., $p'(v) < 0$, $p''(v) > 0$, $b(v)$ is the viscosity, satisfying $b(v) \geq 0$, $b'(v) \leq 0$, and the dispersion coefficient, $d \leq 0$, accounting for capillarity, is constant. We assume that both $p(v)$ and $b(v)$ are smooth (at least C^3).

Two well-known subclasses of (5.1) are the isentropic Navier-Stokes equation, with semi-parabolic (or real) viscosity,

$$
\begin{aligned}
&v_t - u_x = 0, \\
&u_t + p(v)_x = (\frac{u_x}{v})_x,
\end{aligned}
\tag{5.2}
$$

and its less physical counterpart, with parabolic (or artificial) viscosity,

$$
\begin{aligned}
&v_t - u_x = \epsilon v_{xx}, \\
&u_t + p(v)_x = \epsilon u_{xx}.
\end{aligned}
\tag{5.3}
$$

We note that (5.3) can be obtained from (5.1) via Slemrod's transformation [**46**], where $u \to u + \epsilon v_x$ and $v \to v$, $b(v) = 2\epsilon$, and $d = -\epsilon^2$. We further remark that (5.2) and (5.3) are both contained in the Kawashima class, described in Chapter 3.

In this chapter we use energy estimates to examine the spectrum of shock profiles for viscous-dispersive gas dynamics (5.1). Our main result extends the work of Matsumura and Nishihara [**41**] by showing that small-amplitude shocks of (5.1) are spectrally stable.

In addition, we offer a short and novel proof that all monotone shocks, of any amplitude, have no unstable real spectrum. We note that this result is stronger than those given by the Evans function stability index, which only measures the parity of unstable real eigenvalues, see [**1, 10, 53**].

We note that Khodja [**31**] proved zero-mass small-amplitude shock stability for the constant viscosity p-system with the added constant dispersive term u_{xxx}. It appears that the structure of this model is different from the one examined here.

2. Preliminaries

2.1. Shock Profile. stationary solutions of

$$\begin{aligned} &v_t - sv_x - u_x = 0, \\ &u_t - su_x + p(v)_x = (b(v)u_x)_x + dv_{xxx}. \end{aligned} \tag{5.4}$$

Under the rescaling, $x \to -sx$, $t \to s^2 t$ and $u \to -su$, our system takes the form

$$\begin{aligned} &v_t + v_x - u_x = 0, \\ &u_t + u_x + ap(v)_x = (b(v)u_x)_x + dv_{xxx}, \end{aligned} \tag{5.5}$$

where $a = 1/s^2$. Thus, the shock profiles of (5.1) are solutions of the ordinary differential equation

$$v' - u' = 0,$$

$$u' + ap'(v) = (b(v)u')' + dv''',$$

where $(v(\pm\infty), u(\pm\infty)) = (v_\pm, u_\pm)$. This simplifies to

$$v' + ap'(v) = (b(v)v')' + dv'''.$$

By integrating from $-\infty$ to x, we get our profile equation,

$$v - v_- + a(p(v) - p(v_-)) = b(v)v' + dv''. \tag{5.6}$$

We point out that all p-system shocks for an ideal gas are of Lax type [**47**]; without loss of generality, we will assume in this paper that these shocks are Lax-1 shocks, i.e., that $v_+ < v_-$.

REMARK 5.1. *In the absence of capillarity ($d = 0$), the profile equation* (5.6) *is of first order, and thus has a monotone solution. We show below that the solutions in the dispersive case ($d \neq 0$) are also monotone, for sufficiently small-amplitude shocks.*

2.2. Existence and asymptotics. As shown above, the shock profiles of (5.1) reduce to solutions of (5.6) subject to the asymptotically constant boundary conditions, $v(\pm\infty) = v_\pm$. Hence, we can use standard techniques from ordinary differential equations theory to prove existence.

Intuitively, one can see that the zero-diffusion case, $b(v) = 0$ in (5.6), is Hamiltonian, and thus its solution is a conservative nonlinear oscillator. Hence, a positive diffusion term acts as friction and drags the homoclinic orbit toward an asymptotically stable equilibrium.

By writing (5.6) as a first order system, we get

$$\begin{aligned} v' &= w \\ w' &= \frac{1}{d}\left[\phi(v) - b(v)w\right], \end{aligned}$$

where $\phi(v) = v - v_- + a(p(v) - p(v_-))$. Note that $\phi(v) < 0$ between $v_\pm$. The above observation provides us with the Lyapunov function

$$E(v, w) = \frac{1}{2}w^2 + \frac{1}{|d|}\int_{v_-}^{v} \phi(v), \tag{5.7}$$

which is non-negative for $v \in [v_+, v_-]$. It follows that

$$\frac{d}{dx}E(v(x), w(x)) = \nabla E \cdot (v', w') = \frac{b(v)}{|d|}|w|^2 \geq 0. \tag{5.8}$$

Thus, as $x \to -\infty$, bounded orbits are pulled into the minimum $E(v_-, 0) = 0$ of $E(v, w)$. Using this, one can see that there exists a connecting orbit from v_+ to v_-.

We now show that small-amplitude shocks of (5.1) are monotone and follow the same asymptotic limits as the non-dispersive case presented in [**37**],[**42**]. We accomplish this by rescaling and showing, via geometric singular perturbation theory [**6**],[**11**][**25**], that the profile converges smoothly to the non-dispersive case, in the small-amplitude shock limit. Thus, monotonicity of small-amplitude shocks of (5.1) is implied by the monotonicity of the non-dispersive case, as mentioned in Remark 5.1.

We scale according to the amplitude $\epsilon = v_- - v_+$. Let $\bar{v} = (v - v_0)/\epsilon$ and $\bar{x} = \epsilon x$, where $v_0 = v_- - \epsilon/2$. This frame is chosen so that the end-states of the profile are fixed at $\bar{v}_\pm = \pm 1/2$. Additionally, we expand $p(v)$ and $b(v)$ about v_-. Hence

$$\begin{aligned} &\epsilon(\bar{v} - \bar{v}_-)\left(1 + ap(v_-)\right) + \epsilon^2 \frac{ap''(v_-)}{2}(\bar{v} - \bar{v}_-)^2 + \mathcal{O}(\epsilon^3)(\bar{v} - \bar{v}_-)^3 \\ &\quad = \epsilon^2 b(v_-)\bar{v}' + \mathcal{O}(\epsilon^3) b'(v_-)(\bar{v} - \bar{v}_-)\bar{v}' + \epsilon^3 d\bar{v}''. \end{aligned} \tag{5.9}$$

By expanding the Rankine-Hugoniot equality

$$\epsilon = a(p(v_+) - p(v_-)),$$

about v_-, we obtain

$$1 + ap(v_-) = \frac{ap''(v_-)}{2}\epsilon + \mathcal{O}(\epsilon^2). \tag{5.10}$$

Substituting (5.10) into (5.9) and simplifying gives (note $\bar{v}_- = 1/2$)

$$\frac{ap''(v_-)}{2}(\bar{v}^2 - \frac{1}{4}) + \epsilon R(\bar{v}, \bar{v}') = b(v_-)\bar{v}' + \epsilon^3 d\bar{v}''. \tag{5.11}$$

where $R(\bar{v}, \bar{v}') = \mathcal{O}(1)$. Thus, in the $\epsilon = 0$ limit, (5.11) becomes

$$\frac{ap''(v_-)}{2}(\bar{v}^2 - \frac{1}{4}) = b(v_-)\bar{v}', \tag{5.12}$$

which is essentially the same reduction obtained for the viscous Burgers equation. Note that the capillarity term vanishes as well, and thus the reduction is the same as the $(d = 0)$ case.

The slow dynamics of (5.11) take the form

$$v' = w, \tag{5.13a}$$

$$\epsilon d w' = \left[\frac{ap''(v_-)}{2}(\bar{v}^2 - \frac{1}{4}) + \epsilon R(\bar{v}, \bar{v}') - b(v)w\right]. \tag{5.13b}$$

The fast dynamics, obtained by rescaling time $x \to x/\epsilon$, take the form

$$v' = \epsilon w, \tag{5.14a}$$

$$d w' = \left[\frac{ap''(v_-)}{2}(\bar{v}^2 - \frac{1}{4}) + \epsilon R(\bar{v}, \bar{v}') - b(v)w\right]. \tag{5.14b}$$

We can see from the slow dynamics that solutions will remain on the parabola defined by

$$w = \frac{ap''(v_-)}{2b(v_-)}(v^2 - \frac{1}{4}).$$

In addition, from the fast dynamics, we can see that any jumps will be vertical, i.e., $v =$ constant. Thus, no jumps occur since there are no vertical branches, and it follows that small-amplitude shocks approach the solutions for (5.12). Since we assumed our pressure and viscosity laws were at least C^3, it follows that convergence is at least C^2, [**11**]. Hence, for sufficiently small amplitudes, the profiles are monotone. Thus we have proved the following lemma:

LEMMA 5.1. *Small-amplitude shocks of* (5.1) *are monotone.*

We remark that, in the original scale, small-amplitude profiles of (5.1) have the asymptotic properties $|\hat{v}_x| = \mathcal{O}(\epsilon^2)$ and $|\hat{v}_{xx}| = |\hat{v}_x|\mathcal{O}(\epsilon)$, see [**37**],[**42**].

2.3. Linearization. As was done in (2.18), we linearize (5.5) about the profile $(\hat{v}, \hat{u})$ to get the eigenvalue problem

$$\begin{aligned} &\lambda v + v' - u' = 0, \\ &\lambda u + u' + ((ap'(\hat{v}) - b'(\hat{v})\hat{v}_x)v)' = (b(\hat{v})u')' + dv'''. \end{aligned} \tag{5.15}$$

2.4. Essential spectrum. To prove that the essential spectrum is stable, we take the Fourier transform of (5.15) for a constant profile. Hence we get

$$\begin{aligned} &\lambda v + i\xi v - i\xi u = 0, \\ &\lambda u + i\xi u - i\xi c^2 v + \xi^2 bu + i\xi^3 dv = 0, \end{aligned}$$

where $-c^2 = ap'(\hat{u})$, b, d are all constant. By taking taking the inner product with v and u respectively, we get

$$\lambda\|v\|^2 + i\xi\|v\|^2 - i\xi\langle v, u\rangle = 0, \tag{5.16a}$$

$$\lambda\|u\|^2 + i\xi\|u\|^2 - i\xi\left(c^2 - \xi^2 d\right)\langle u, v\rangle + \xi^2 b\|u\|^2 = 0. \tag{5.16b}$$

By substituting appropriately and taking the real part, we arrive at

$$\Re e\lambda\|u\|^2 + \xi^2 b\|u\|^2 + \Re e\lambda\left(c^2 - \xi^2 d\right)\|v\|^2 = 0 \tag{5.17}$$

Hence we see that $\Re e\lambda(\xi) < 0$ when $\xi \neq 0$, and $\Re e\lambda(\xi) = 0$ when $\xi = 0$.

2.5. Integrated eigenvalue problem. Following Lemma 2.2, we have the integrated eigenvalue problem

$$\lambda v + v' - u' = 0, \tag{5.18a}$$

$$\lambda u + u' + (ap'(\hat{v}) - b'(\hat{v})\hat{v}_x)v' = b(\hat{v})u'' + dv''' \tag{5.18b}$$

3. Spectral stability of small-amplitude shocks

THEOREM 5.1. *Small-amplitude shocks of* (5.1) *are spectrally stable.*

PROOF. By the previous lemma, we can assume that $\hat{v}_x < 0$. Let $f(\hat{v}) = b'(\hat{v})\hat{v}_x - ap'(\hat{v})$. We note that for small-amplitude shocks, $f(\hat{v}) > 0$ and $f'(\hat{v}) < 0$. By multiplying (5.18b) by the conjugate $\bar{u}/f(\hat{v})$ and integrating from ∞ to $-\infty$, we have

$$\int_{-\infty}^{+\infty} \frac{\lambda u\bar{u}}{f(\hat{v})} + \int_{-\infty}^{+\infty} \frac{u'\bar{u}}{f(\hat{v})} - \int_{-\infty}^{+\infty} v'\bar{u} = \int_{-\infty}^{+\infty} \frac{b(\hat{v})u''\bar{u}}{f(\hat{v})} + \int_{-\infty}^{+\infty} \frac{dv'''\bar{u}}{f(\hat{v})}.$$

Integrating the last three terms by parts and appropriately using (5.18a) to substitute for u' in the third term gives us

$$\begin{aligned}\int_{-\infty}^{+\infty} \frac{\lambda|u|^2}{f(\hat{v})} &+ \int_{-\infty}^{+\infty} \frac{u'\bar{u}}{f(\hat{v})} + \int_{-\infty}^{+\infty} v(\overline{\lambda v + v'}) + \int_{-\infty}^{+\infty} \frac{b(\hat{v})}{f(\hat{v})}|u'|^2 \\ &= -\int_{-\infty}^{+\infty} \left(\frac{b(\hat{v})}{f(\hat{v})}\right)' u'\bar{u} - d\int_{-\infty}^{+\infty} \frac{1}{f(\hat{v})}v''\bar{u}' - d\int_{-\infty}^{+\infty} \left(\frac{1}{f(\hat{v})}\right)' v''\bar{u}.\end{aligned}$$

We take the real part and appropriately integrate by parts:

$$\begin{aligned}\Re e(\lambda) \int_{-\infty}^{+\infty} \left[\frac{|u|^2}{f(\hat{v})} + |v|^2\right] &- \frac{1}{2}\int_{-\infty}^{+\infty} \left(\frac{1}{f(\hat{v})}\right)' |u|^2 + \int_{-\infty}^{+\infty} \frac{b(\hat{v})}{f(\hat{v})}|u'|^2 \\ &= \frac{1}{2}\int_{-\infty}^{+\infty} \left(\frac{b(\hat{v})}{f(\hat{v})}\right)'' |u|^2 - d\,\Re e\left[\int_{-\infty}^{+\infty} \frac{1}{f(\hat{v})}v''\bar{u}' + \int_{-\infty}^{+\infty} \left(\frac{1}{f(\hat{v})}\right)' v''\bar{u}\right].\end{aligned}$$

Thus, by integrating the last two terms by parts and further simplifying, for $\lambda \geq 0$, we have

$$\begin{aligned}-\frac{1}{2}\int_{-\infty}^{+\infty} \left(\frac{1}{f(\hat{v})}\right)' |u|^2 &+ \int_{-\infty}^{+\infty} \frac{b(\hat{v})}{f(\hat{v})}|u'|^2 \leq \frac{1}{2}\int_{-\infty}^{+\infty} \left(\frac{b(\hat{v})}{f(\hat{v})}\right)'' |u|^2 \\ &+ d\,\Re e\left[\int_{-\infty}^{+\infty} \frac{1}{f(\hat{v})}v'\bar{u}'' + 2\int_{-\infty}^{+\infty} \left(\frac{1}{f(\hat{v})}\right)' v'\bar{u}' + \int_{-\infty}^{+\infty} \left(\frac{1}{f(\hat{v})}\right)'' v'\bar{u}\right].\end{aligned}$$

Repeating the above steps again gives,

$$\begin{aligned}-\frac{1}{2}\int_{-\infty}^{+\infty} \left(\frac{1}{f(\hat{v})}\right)' |u|^2 &+ \int_{-\infty}^{+\infty} \frac{b(\hat{v})}{f(\hat{v})}|u'|^2 + \frac{d}{2}\int_{-\infty}^{+\infty} \left(\frac{1}{f(\hat{v})}\right)' |v'|^2 \\ &\leq \frac{1}{2}\int_{-\infty}^{+\infty} \left(\frac{b(\hat{v})}{f(\hat{v})}\right)'' |u|^2 + d\,\Re e\left[2\int_{-\infty}^{+\infty} \left(\frac{1}{f(\hat{v})}\right)' v'\bar{u}' + \int_{-\infty}^{+\infty} \left(\frac{1}{f(\hat{v})}\right)'' v'\bar{u}\right].\end{aligned}$$

We note that since both $d \leq 0$ and $\hat{v}_x < 0$, then all the terms on the left-hand side are non-negative. Moreover, since $|\hat{v}_x| = \mathcal{O}(\epsilon^2)$ and $|\hat{v}_{xx}| = |\hat{v}_x|\mathcal{O}(\epsilon)$, it follows that the right-hand side of the above equation is bounded above by

$$C_1 \int_{-\infty}^{+\infty} \epsilon|\hat{v}_x||u|^2 + 2d \int_{-\infty}^{+\infty} \left(\frac{1}{f(\hat{v})}\right)' |v'||u'| + C_2 \int_{-\infty}^{+\infty} \epsilon|\hat{v}_x||v'||u|.$$

Thus, by Young's inequality, we have

$$\begin{aligned} & -\frac{1}{2}\int_{-\infty}^{+\infty} \left(\frac{1}{f(\hat{v})}\right)' |u|^2 + \int_{-\infty}^{+\infty} \frac{b(\hat{v})}{f(\hat{v})}|u'|^2 + \frac{d}{2}\int_{-\infty}^{+\infty} \left(\frac{1}{f(\hat{v})}\right)' |v'|^2 \\ & < C_1 \int_{-\infty}^{+\infty} \epsilon|\hat{v}_x||u|^2 + 2d \int_{-\infty}^{+\infty} \left(\frac{1}{f(\hat{v})}\right)' \left[\frac{|v'|^2}{4\eta_1} + \eta_1|u'|^2\right] \\ & \quad + C_2 \int_{-\infty}^{+\infty} \epsilon|\hat{v}_x| \left[\frac{|v'|^2}{4\eta_2} + \eta_2|u'|^2\right]. \end{aligned} \tag{5.19}$$

We can see that for $\eta_1 > 1$ and η_2, ϵ sufficiently small, the left side dominates the right side, which is a contradiction when $\Re e\lambda \geq 0$. $\square$

4. Monotone large-amplitude shocks

THEOREM 5.2. *Monotone shocks of* (5.1) *have no unstable real spectrum.*

PROOF. As in the previous proof, we assume that $\hat{v}_x < 0$. Then multiply (5.18b) by the conjugate $\bar{v}$ and integrate from ∞ to $-\infty$. This gives

$$\begin{aligned} \int_{-\infty}^{+\infty} \lambda u\bar{v} + \int_{-\infty}^{+\infty} u'\bar{v} + \int_{-\infty}^{+\infty} (ap'(\hat{v}) - a'(\hat{v})\hat{v}_x)v'\bar{v} \\ = \int_{-\infty}^{+\infty} b(\hat{v})u''\bar{v} + d\int_{-\infty}^{+\infty} v'''\bar{v}. \end{aligned}$$

Notice that on the real line, $\bar{\lambda} = \lambda$. Thus, we have

$$\begin{aligned} \int_{-\infty}^{+\infty} \bar{\lambda} u\bar{v} + \int_{-\infty}^{+\infty} u'\bar{v} + \int_{-\infty}^{+\infty} (ap'(\hat{v}) - b'(\hat{v})\hat{v}_x)v'\bar{v} \\ = \int_{-\infty}^{+\infty} b(\hat{v})u''\bar{v} - d\int_{-\infty}^{+\infty} v''\bar{v}'. \end{aligned}$$

Using (5.18a) to substitute for $\bar{\lambda}\bar{v}$ in the first term and for u'' in the last term, we get

$$\begin{aligned} \int_{-\infty}^{+\infty} u(\bar{u}' - \bar{v}') + \int_{-\infty}^{+\infty} u'\bar{v} + \int_{-\infty}^{+\infty} (ap'(\hat{v}) - b'(\hat{v})\hat{v}_x)v'\bar{v} \\ = \int_{-\infty}^{+\infty} b(\hat{v})(\lambda v' + v'')\bar{v} - d\int_{-\infty}^{+\infty} v''\bar{v}'. \end{aligned}$$

Separating terms and simplifying gives

$$\int_{-\infty}^{+\infty} u\bar{u}' + 2\int_{-\infty}^{+\infty} u'\bar{v} + \int_{-\infty}^{+\infty} (ap'(\hat{v}) - b'(\hat{v})\hat{v}_x)v'\bar{v}$$
$$= \lambda \int_{-\infty}^{+\infty} b(\hat{v})v'\bar{v} + \int_{-\infty}^{+\infty} b(\hat{v})v''\bar{v} - d\int_{-\infty}^{+\infty} v''\bar{v}'.$$

We further simplify by substituting for u' in the second term and integrating the last terms by parts to give,

$$\int_{-\infty}^{+\infty} u\bar{u}' + 2\int_{-\infty}^{+\infty} (\lambda v + v')\bar{v} + \int_{-\infty}^{+\infty} (ap'(\hat{v}) - b'(\hat{v})\hat{v}_x)v'\bar{v}$$
$$= \lambda \int_{-\infty}^{+\infty} b(\hat{v})v'\bar{v} - \int_{-\infty}^{+\infty} b'(\hat{v})\hat{v}_x v'\bar{v} - \int_{-\infty}^{+\infty} b(\hat{v})|v|^2 - d\int_{-\infty}^{+\infty} v''\bar{v}',$$

which yields

$$\int_{-\infty}^{+\infty} u\bar{u}' + 2\lambda\int_{-\infty}^{+\infty} |v|^2 + 2\int_{-\infty}^{+\infty} v'v + a\int_{-\infty}^{+\infty} p'(\hat{v})v'\bar{v} + \int_{-\infty}^{+\infty} a(\hat{v})|v|^2$$
$$= \lambda \int_{-\infty}^{+\infty} b(\hat{v})v'\bar{v} - d\int_{-\infty}^{+\infty} v''\bar{v}'.$$

By taking the real part (recall that $\lambda \in \mathbf{R}$), we arrive at

$$2\lambda \int_{-\infty}^{+\infty} |v|^2 - \frac{a}{2}\int_{-\infty}^{+\infty} p''(\hat{v})\hat{v}_x|v|^2 + \int_{-\infty}^{+\infty} b(\hat{v})|v|^2$$
$$+ \frac{\lambda}{2}\int_{-\infty}^{+\infty} b'(\hat{v})\hat{v}_x|v|^2 = 0.$$

This is a contradiction when $\lambda \geq 0$. Thus, there are no positive real eigenvalues for (5.5). $\square$

REMARK 5.2. *It is not known whether or not the point spectrum can be complex for profiles for* (5.1)*, or more generally, systems with an entropy. This is an interesting question for further investigation.*

APPENDIX A

Notes on the Kawashima class

In this appendix, we discuss the structure built into the Kawashima class and prove the key results given in Theorem 3.1. Before doing this, however, we need some facts from linear algebra.

1. Linear algebra

Let $M_n(\mathbb{C})$ denote the set of $n \times n$ matrices over $\mathbb{C}$ with inner product

$$\langle X, Y \rangle = \mathrm{Tr}(XY^*).$$

Definition A.1. *Given $A \in M_n(\mathbb{C})$, we define the Ad operator on $M_n(\mathbb{C})$ as $Ad_A(X) = [A, X]$. We note that Ad_A is a derivation on $M_n(\mathbb{C})$, i.e., both linearity and the Leibniz rule hold*[1] *and that Ad_A is always singular since A commutes with any admissible function of itself, e.g., I, A^k, etc. The following lemmas further illustrate important properties of the Ad operator:*

Lemma A.1. *Let $A \in M_n(\mathbb{C})$. The following hold:*

(i). $(Ad_A)^* = Ad_{A^*}$.

(ii). *If A is normal, then Ad_A is normal.*

(iii). *If A is Hermitian, then Ad_A is Hermitian.*

(iv). *If A is semi-simple, with n eigenvalues μ_j and corresponding right and left eigenvectors r_j and l_j, respectively, then the n^2 eigenvalues of Ad_A are $\mu_j - \mu_k$ corresponding to the eigenvectors $X_{jk} = r_j l_k$, respectively.*

[1] The *Ad* operator has roots in Lie theory, where numerous other important properties are developed.

PROOF. We get the following by direct calculation:

(i). We show that $(\mathrm{Ad}_A)^* = \mathrm{Ad}_{A^*}$.

$$\begin{aligned}\langle X, \mathrm{Ad}_A^* Y\rangle &= \langle \mathrm{Ad}_A X, Y\rangle \\ &= \mathrm{Tr}((AX - XA)Y^*) \\ &= \mathrm{Tr}(X(Y^*A - AY^*)) \\ &= \mathrm{Tr}(X(A^*Y - YA^*)^*) \\ &= \langle X, \mathrm{Ad}_{A^*} Y\rangle.\end{aligned}$$

(ii). We show that $\mathrm{Ad}_{A^*}\mathrm{Ad}_A = \mathrm{Ad}_A\mathrm{Ad}_{A^*}$.

$$\begin{aligned}\mathrm{Ad}_{A^*}(\mathrm{Ad}_A X) &= [A^*, AX - XA] \\ &= A^*AX - A^*XA - AXA^* + XAA^* \\ &= AA^*X - A^*XA - AXA^* + XA^*A \\ &= A(A^*X - XA^*) - (A^*X - XA^*)A \\ &= [A, A^*X - XA^*] \\ &= \mathrm{Ad}_A(\mathrm{Ad}_{A^*} X).\end{aligned}$$

(iii). From (i) we have $(\mathrm{Ad}_A)^* = \mathrm{Ad}_{A^*} = \mathrm{Ad}_A$.

(iv). We show that $\mathrm{Ad}_A X_{jk} = (\mu_j - \mu_k)X_{jk}$.

$$\begin{aligned}\mathrm{Ad}_A X_{jk} &= AX_{jk} - X_{jk}A \\ &= Ar_jl_k - r_jl_kA \\ &= \mu_jr_jl_k - r_jl_k\mu_k \\ &= (\mu_j - \mu_k)X_{jk}.\end{aligned}$$

$\square$

LEMMA A.2. *Let $A \in M_n(\mathbb{C})$ be semi-simple and suppose that $\mu_1 P_1 + \ldots + \mu_r P_r$ is its spectral resolution, where $\mu_1, \ldots, \mu_r$ are the distinct eigenvalues of A corresponding to the eigenprojections $P_1, \ldots, P_r$, respectively. We define the following linear operator:*

$$\Pi_A(X) = \sum_{j=1}^{r} P_j X P_j, \quad X \in M_n(\mathbb{C}). \tag{A.1}$$

Then the following hold:

(i). $\Pi_A(X)$ *is a projection.*

(ii). $\mathcal{R}(\Pi_A) = \mathcal{N}(Ad_A)$.

(iii). *If A is normal, then Π_A is an orthonormal projection onto $\mathcal{N}(Ad_A)$.*

PROOF. We get the following by direct calculation:

(i). We show that $\Pi_A \Pi_A = \Pi_A$.

$$\begin{aligned}
\Pi_A(\Pi_A(X)) &= \sum_{j=1}^{r} P_j \Pi_A(X) P_j = \sum_{j,k=1}^{r} P_j P_k X P_k P_j \\
&= \sum_{j,k=1}^{r} P_j \delta_{jk} X P_j \delta_{jk} \\
&= \sum_{j=1}^{r} P_j X P_j \\
&= \Pi_A(X).
\end{aligned}$$

(ii). To show that $\mathcal{R}(\Pi_A) \subset \mathcal{N}(\mathrm{Ad}_A)$, let $X \in M_n(\mathbb{C})$ and note that $AP_j = P_jA = \mu_j P_j$. Then

$$\begin{aligned} A\Pi_A(X) &= \sum_{j=1}^{r} AP_jXP_j \\ &= \sum_{j=1}^{r} \mu_j P_jXP_j \\ &= \sum_{j=1}^{r} P_jXP_jA \\ &= \Pi_A(X)A. \end{aligned}$$

Conversely, if $X \in \mathcal{N}(\mathrm{Ad}_A)$, then the spectral resolution of X has the same eigenprojections P_j as A, i.e., $X = \lambda_1 P_1 + \ldots + \lambda_r P_r$, where $XP_j = P_jX$. Thus

$$\Pi_A(X) = \sum_{j=1}^{r} P_jXP_j = \sum_{j=1}^{r} XP_jP_j = X\sum_{j=1}^{r} P_j = X,$$

which proves that $\mathcal{N}(\mathrm{Ad}_A) \subset \mathcal{R}(\Pi_A)$.

(iii). Since A is normal then $P_j = P_j^*$. It suffices to show that Π_A is Hermitian.

$$\begin{aligned} \langle X, \Pi_A^*(Y)\rangle &= \langle \Pi_A(X), Y\rangle \\ &= \mathrm{Tr}(\sum_{j=1}^{r} P_jXP_jY^*) \\ &= \mathrm{Tr}(X\sum_{j=1}^{r} P_jY^*P_j) \\ &= \mathrm{Tr}(X\Pi_A(Y)^*) \\ &= \langle X, \Pi_A(Y)\rangle. \end{aligned}$$

□

REMARK A.1. *If A is normal, then by the uniqueness of the orthogonal projection, $\mathcal{N}(\Pi_A) = \mathcal{R}(Ad_A)$.*

LEMMA A.3. *Let A be normal.*

(i). *If B is Hermitian then so is $\Pi_A(B)$.*

(ii). *If B is nonnegative definite, then so is $\Pi_A(B)$.*

PROOF. Since A is normal, $P^* = P$ Hence:

(i). We show that $\Pi_A(B)^* = \Pi_A(B)$.

$$\Pi_A(B)^* = \sum_{j=1}^{r}(P_j B P_j)^* = \sum_{j=1}^{r} P_j B P_j = \Pi_A(B).$$

(ii). We show that $\Pi_A(B) \geq 0$

$$\langle X, \Pi_A(B)X\rangle = \sum_{j=1}^{r}\langle X, P_j B P_j X\rangle = \sum_{j=1}^{r}\langle P_j X, B P_j X\rangle \geq 0.$$

□

LEMMA A.4. *Let A be normal and $B \in M_n(\mathbb{C})$. Then there exists $K \in M_n(\mathbb{C})$ such that*

$$B = \Pi_A(B) + [A, K]. \tag{A.2}$$

In addition, we have:

(i). *If A and B are Hermitian, then K can be chosen to be skew-Hermitian.*

(ii). *If A and B are real, then K can be chosen to be real also.*

PROOF. By Lemma A.2, given $B \in M_n(\mathbb{C})$ there is a unique decomposition $B = B_1 + B_2$, where $B_1 \in \mathcal{R}(\Pi_A)$ and $B_2 \in \mathcal{R}(\mathrm{Ad}_A)$. Since Π_A is a projection, then $B_1 = \Pi_A(B)$. Finally, if $B_2 \in \mathcal{R}(\mathrm{Ad}_A)$, then there exists some $K \in M_n(\mathbb{C})$ such that $B_2 = [A, K]$. Hence (A.2) holds.

(i). Suppose B is Hermitian, by Lemma A.3, we know that $B_1 = \Pi_A(B)$ is also Hermitian, and hence so is $[A, K]$. Since A is Hermitian,

$$[A, K] = [A, K]^* = K^*A - AK^* = [K^*, A] = -[A, K^*]. \tag{A.3}$$

Setting $K = K_1 + K_2$, where $K_1 = (K + K^*)/2$ and $K_2 = (K - K^*)/2$, we see from (A.3) that $[A, K] = [A, K_2]$ and $[A, K_1] = 0$. Hence K_2 can be used in place of K in (A.2).

(ii). We use a similar argument. Suppose B is real. Then $\Pi_A(B)$ is real, and hence $[A, K]$ is also. Hence,

$$[A, K] = \overline{[A, K]} = [A, \bar{K}]. \tag{A.4}$$

Setting $K = K_1 + K2$, where $(K + \bar{K})/2$ and $K_2 = (K - \bar{K})/2$, we see from (A.4) that $[A, K_2] = 0$, and hence K_1 can be used in place of K in (A.2).

□

THEOREM A.1. *Suppose that A and B are real Hermitian matrices and that B is nonnegative definite. Then the following are equivalent:*

(i). *There exists a real skew-symmetric matrix K such that $B+[K, A]$ is positive definite.*

(ii). *No eigenvector of A lies in the kernel of B.*

PROOF. $(i) \Rightarrow (ii)$: Assume (i) holds and suppose that ψ is an eigenvector of A with eigenvalue μ, such that $\psi \in \mathcal{N}(B)$. We arrive at the following contradiction:

$$\begin{aligned} 0 &< \langle \psi, (B + [K, A])\psi \rangle \\ &= \langle \psi, KA\psi \rangle - \langle \psi, AK\psi \rangle \\ &= 2\langle \psi, KA\psi \rangle \\ &= 2\mu \langle \psi, K\psi \rangle = 0. \end{aligned}$$

$(ii) \Rightarrow (i)$: By Lemma A.3, we have that $\Pi_A(B) = B + [K, A]$ is real symmetric and nonnegative definite. It suffices to show strict definiteness, i.e., $\langle x, \Pi_A(B)x \rangle = 0$

implies $x = 0$. Using the spectral resolution $A = \mu_1 P_1 + \ldots + \mu_r P_r$, we have

$$\begin{aligned} 0 &= \langle x, \Pi_A(B)x \rangle \\ &= \sum_{j=1}^{r} \langle P_j x, B P_j x \rangle. \end{aligned}$$

Hence $\forall j$, $x_j = P_j x \in \mathcal{N}(B)$. However each x_j is an eigenvalue of A, which implies from (ii) that $x_j = 0 \, \forall j$. It follows that $x = \sum_{j=1}^{r} P_j x = \sum_{j=1}^{r} x_j = 0$. $\square$

2. Admissibility Theorem

Consider the linear one-dimensional system

$$v_t = Lv := -Av_x + Bv_{xx} - Dv, \tag{A.5}$$

where A, B, D are real symmetric constant matrices and B, D are nonnegative definite. Recall that since this is a constant coefficient system, the spectrum is all essential, i.e., $\sigma(L) = \sigma_e(L)$. By taking the Fourier transform, we see that $\lambda \in \sigma_e(L)$ if there exists a non-trivial v such that

$$\{\lambda I + i\xi A + \xi^2 B + D\, \}v = 0, \tag{A.6}$$

where $\xi \in \mathbb{R}$. Said differently,

$$\lambda \in \sigma_e(L) \quad \text{iff} \quad \lambda \in \sigma(-i\xi A - \xi^2 B - D). \tag{A.7}$$

Thus, the essential spectrum for (A.5) is given by the n curves $\lambda_j(\xi)$ in (A.7). Hence, the matrices A, B, D completely characterize the essential spectrum.

In the previous section, we proved some important facts from linear algebra. We can apply Theorem A.1 to (A.7). In particular, if (A.5) is genuinely coupled, i.e., no eigenvector of A lies in $\mathcal{N}(B) \cap \mathcal{N}(D)$, then there exists a real skew-symmetric $K \in M_n(\mathbb{C})$ such that $\Re e(KA) + B + D$ is positive definite. The existence of K gives us the following lemma:

LEMMA A.5. *Suppose for matrices A, B, D, given by* (A.5), *there exists a real skew symmetric $K \in M_n(\mathbb{C})$ such that $\Re e(KA) + B + D$ is positive definite. Then for some $\theta > 0$ we have that the essential spectrum $\lambda(\xi)$ satisfies*

$$\Re e\lambda(\xi) \leq -\theta|\xi|^2/(1+|\xi|^2). \tag{A.8}$$

PROOF. We combine two spectral energy estimates: First, by taking the inner product of (A.6) with v, we get the following standard Friedrich's-type estimate [**6**]:

$$\begin{aligned} 0 =& \langle v, \lambda v\rangle + \langle v, i\xi Av\rangle + \langle v, \xi^2 Bv\rangle + \langle v, Dv\rangle \\ =& \lambda\|v\| + i\xi\langle v, Av\rangle + \xi^2\langle v, Bv\rangle + \langle v, Dv\rangle. \end{aligned}$$

Taking the real part yields

$$0 = \Re e\lambda\|v\|^2 + |\xi|^2\langle v, Bv\rangle + \langle v, Dv\rangle.$$

Multiplying by $1 + |\xi|^2$ and noting for some $\theta_1, \theta_2 > 0$, that

$$\langle v, Bv\rangle \geq \theta_1\|Bv\|^2 \quad \text{and} \quad \langle v, Dv\rangle \geq \theta_2\|Dv\|^2,$$

we have

$$\Re e\lambda(1+|\xi|^2)\|v\|^2 + \theta_1|\xi|^4\|Bv\|^2 + \theta_2\|Dv\|^2 + |\xi|^2\left[\langle v, Bv\rangle + \langle v, Dv\rangle\right] \leq 0. \tag{A.9}$$

For the second estimate, we multiply the skew-symmetric matrix K by (A.6) and take the inner product with $i\xi v$ to get

$$\begin{aligned} 0 =& \langle i\xi v, \lambda Kv\rangle + \langle i\xi v, i\xi KAv\rangle + \langle i\xi v, \xi^2 KBv\rangle + \langle i\xi v, KDv\rangle \\ =& -i\xi\lambda\langle v, Kv\rangle + |\xi|^2\langle v, KAv\rangle - i\xi^3\langle v, KBv\rangle - i\xi\langle v, KDv\rangle. \end{aligned}$$

By taking the real part and using Young's inequality, we get

$$\begin{aligned} \frac{|\xi|^2}{2}\langle v, [K, A]v\rangle \leq& \Re e\lambda|\xi|\|K\|\|v\|^2 + |\xi|^3\|v\|\|K\|\|Bv\| + |\xi|\|v\|\|K\|\|Dv\| \\ \leq& \frac{\|K\|}{2}\Re e\lambda(1+|\xi|^2)\|v\|^2 + |\xi|^2\left(\epsilon_1\|v\|^2 + \frac{|\xi|^2\|K\|^2}{4\epsilon_1}\|Bv\|^2\right) \\ &+ \left(\epsilon_2|\xi|^2\|v\|^2 + \frac{\|K\|^2}{4\epsilon_2}\|Dv\|^2\right). \end{aligned} \tag{A.10}$$

By choosing $\epsilon_1 + \epsilon_2 \leq \theta$, where $\theta > 0$ satisfies $[K, A] + B + D \geq 2\theta \cdot I$, and setting

$$M = \max\{\frac{\|K\|^2}{4\epsilon_1\theta_1}, \frac{\|K\|^2}{4\epsilon_2\theta_2}, \frac{\|K\|}{2} + 1\},$$

we have

$$\frac{|\xi|^2}{2}\langle v, [K, A]v\rangle \leq \Re e\lambda(M-1)(1+|\xi|^2)\|v\|^2 + M\theta_1|\xi|^4\|Bv\|^2 + M\theta_2\|Dv\|^2 + \theta|\xi|^2\|v\|^2.$$

Adding to $M\times$ (A.9) yields

$$\Re e\lambda(1 + |\xi|^2)\|v\|^2 + \theta|\xi|^2\|v\|^2 \leq 0.$$

□

Remark A.2. *The operator L is said to be strictly dissipative if $\Re e\lambda(\xi) < 0$ for each $\xi \in \mathbb{R} \setminus \{0\}$. Hence, the above lemma relates genuine coupling and strict dissipativity. We summarize this section with the following theorem, which shows that for our system of interest, they are equivalent.*

Theorem A.2. *Consider the operator equation $v_t = Lv$ given in* (A.5)*. Let $\lambda(\xi)$ be the value λ of the nontrivial solution ϕ of* (A.6)*. Then the following are equivalent:*

(i). *L is strictly dissipative.*

(ii). *L is genuinely coupled, i.e., no eigenvector ϕ of A is in $\mathcal{N}(B) \cap \mathcal{N}(D)$.*

(iii). *There exists a real skew-symmetric $K \in M_n(\mathbb{C})$ such that $\Re\mathrm{e}(KA) + B + D$ is positive definite.*

(iv). *There exists $\theta > 0$ such that $\Re e\lambda(\xi) \leq -\theta|\xi|^2/(1 + |\xi|^2)$.*

Proof. $(i) \Rightarrow (ii)$: Suppose ϕ satisfies both $A\phi = \mu\phi$ and $\phi \in \mathcal{N}(B) \cap \mathcal{N}(D)$. Then

$$(i\xi A + \xi^2 B + D)\phi = i\xi\mu\phi.$$

Hence, $\Re e\lambda(\xi) = \Re\mathrm{e}(-i\xi\mu) = 0$, which contradicts (i). $(ii) \Rightarrow (iii)$: Proven in Theorem A.1. $(iii) \Rightarrow (iv)$: Proven in Lemma A.5. $(iv) \Rightarrow (i)$: Trivial. □

3. The Kawashima Class

We conclude this Appendix by stating in full generality, the Kawashima class, which equates genuinely coupled symmetrizable systems with strict dissipativity and also the existence of a skew symmetric multiplier which is crucial in the derivative estimate of Lemma 3.2.

Consider a one-dimensional system

$$u_t + f(u)_x - (B(u)u_x)_x + (C(u)u_{xx})_x + Q(u) = 0,$$

where $x \in \mathbb{R}$, $u, f \in \mathbb{R}^n$, and $B, C, Q \in \mathbb{R}^{n\times n}$ are all twice continuously differentiable. Moreover, in some neighborhood $\mathcal{U}$ of a particular base point u_*, i.e., the following assumptions hold:

Assumption A.1 (Symmetrizability). *For all $u \in \mathcal{U}$, there exists a symmetrizer $A^0(u)$, symmetric and positive definite, such that the terms $A^0(u)df(u)$, $A^0(u)B(u)$, and $A^0(u)dQ(u)$ are all symmetric, and both $A^0(u)B(u)$ and $A^0(u)dQ(u)$ are nonnegative definite.*

Assumption A.2 (Genuine coupling). *For $u \in \mathcal{U}$, there is no eigenvector of $df(u)$ lying in $\mathcal{N}(B(u)) \cap \mathcal{N}(dQ(u))$.*

Assumption A.3 (Block structure). *The left kernels of $B(u)$ and $dQ(u)$ are independent of u.*

APPENDIX B

Notes on the Jin-Xin relaxation model

In this appendix, we examine the structure of the Jin-Xin relaxation model as it relates to the Kawashima class. We present the following theorem:

THEOREM B.1. *Given that Assumption 4.2 holds, we have that Assumption 4.1 holds if and only if* $[A, df] = 0$.

PROOF. Suppose Assumption 4.1 holds. Then in in some neighborhood $\mathcal{U}$ of a particular base point (u_*, v_*),

$$A^0 = \begin{pmatrix} \alpha & \beta \\ \beta^T & \gamma \end{pmatrix}, \tag{B.1}$$

is symmetric and positive definite, where α, β, and γ are $n \times n$ matrix-valued functions. Hence, α, γ are symmetric, positive-definite. In addition, we have that

$$\begin{pmatrix} \alpha & \beta \\ \beta^T & \gamma \end{pmatrix} \begin{pmatrix} 0 & I \\ A & 0 \end{pmatrix} = \begin{pmatrix} \beta A & \alpha \\ \gamma A & \beta^T \end{pmatrix}$$

is symmetric, and

$$\begin{pmatrix} \alpha & \beta \\ \beta^T & \gamma \end{pmatrix} \begin{pmatrix} 0 & 0 \\ df & -I \end{pmatrix} = \begin{pmatrix} \beta df & -\beta \\ \gamma df & -\gamma \end{pmatrix}$$

is symmetric and positive definite. Hence, it follows that β, βA, and βdf must all be symmetric. In addition, $\alpha = \alpha^T = \gamma A$ and $\beta = -\gamma df$. Hence,

$$A^0 = \begin{pmatrix} \gamma A & -\gamma df \\ -\gamma df & \gamma \end{pmatrix} = \begin{pmatrix} \gamma & 0 \\ 0 & \gamma \end{pmatrix} \begin{pmatrix} A & -df \\ -df & I \end{pmatrix}. \tag{B.2}$$

Since γ is positive-definite, it follows that γ is invertible. Moreover, we have that $\beta A = -\gamma df A$ is symmetric and hence

$$\gamma df A = (\gamma df A)^T = A^T df^T \gamma = A^T \gamma df = \gamma A df.$$

Therefore, since γ is invertible, commutation follows, $df A = A df$. We remark that by (B.2), every symmetrizer can be uniquely determined by γ.

Conversely, suppose that $[A, df] = 0$ in some neighborhood $\mathcal{U}$ of a particular base point (u_*, v_*), there exists a matrix-valued function $S = S(U,V)$ such that SAS^{-1} and $Sdf S^{-1}$ are both diagonal. Thus, we let $\gamma = S^T S$ and show that

$$A^0 = \begin{pmatrix} \gamma & 0 \\ 0 & \gamma \end{pmatrix} \begin{pmatrix} A & -df \\ -df & I \end{pmatrix},$$

is a symmetrizer for (4.10). Note that γ is symmetric and positive definite. Left multiplying A^0 to (4.10) gives

$$\begin{aligned} \lambda\, A^0 \begin{pmatrix} U \\ V \end{pmatrix} + \begin{pmatrix} -s\gamma A - \gamma df A & -\gamma A + s\gamma df \\ -s\gamma df - \gamma A & \gamma df - s\,\gamma \end{pmatrix} \begin{pmatrix} U \\ V \end{pmatrix}_x \\ = \begin{pmatrix} -\gamma df^2 & \gamma df \\ \gamma df & -\gamma \end{pmatrix} \begin{pmatrix} U \\ V \end{pmatrix}. \end{aligned} \tag{B.3}$$

Thus, it suffices to show that γA, γdf, $\gamma df A$, and γdf^2 are all symmetric and that A^0 is positive definite. Note that since SAS^{-1} and $Sdf S^{-1}$ are diagonal, it follows that $SAS^{-1} = (S^{-1})^T A^T S^T$ and $Sdf S^{-1} = (S^{-1})^T df^T S^T$. Thus,

$$\begin{aligned} (\gamma A)^T =& (S^T S A)^T \\ =& (S^T (SAS^{-1}) S)^T \\ =& (S^T ((S^{-1})^T A^T S^T) S)^T \\ =& (A^T \gamma)^T \\ =& \gamma A. \end{aligned}$$

The others follow similarly. Hence, A^0 is symmetric and makes (B.3) symmetric as well. Finally, we show that A^0 is positive: Let

$$y = \begin{pmatrix} S & 0 \\ 0 & S \end{pmatrix} x.$$

Then

$$\begin{aligned} \langle x, A^0 x\rangle &= \langle x, \begin{pmatrix} S^T S & 0 \\ 0 & S^T S \end{pmatrix} \begin{pmatrix} A & -df \\ -df & I \end{pmatrix} x\rangle \\ &= \langle y, \begin{pmatrix} S & 0 \\ 0 & S \end{pmatrix} \begin{pmatrix} A & -df \\ -df & I \end{pmatrix} \begin{pmatrix} S^{-1} & 0 \\ 0 & S^{-1} \end{pmatrix} y\rangle \\ &= \langle y, \begin{pmatrix} \tilde{A} & -\Lambda \\ -\Lambda & I \end{pmatrix} y\rangle, \end{aligned}$$

where $\tilde{A} = SAS^{-1}$ and $\Lambda = S\Lambda S^{-1}$ are both diagonal. Thus, by reordering the coordinates of y, we can write the matrix as block diagonals of the form

$$\begin{pmatrix} \tilde{A}_i & -\Lambda_i \\ -\Lambda_i & 1 \end{pmatrix}.$$

Hence, we have positivity if and only if each block is positive. However, a 2×2 block is positive if the trace and determinant are both positive, which follows from the fact that $\tilde{A}$ is positive and from Assumption 4.2, that $\tilde{A} - \Lambda^2 > 0$. $\square$

APPENDIX C

Identities for Inner Products

For convenience of the reader, we give here the elementary computation that plays in the spectral, complex-valued context the role played by Friedrich's-type estimates for real-valued time-evolutionary systems with symmetric coefficients [**7**]. Hereafter, let $\|\cdot\|$, $\langle\cdot,\cdot\rangle$ denote the standard complex L^2 norm and inner product, $|\cdot|$ and "$\cdot$" the complex vector norm and inner product, and $\int f$ the integral $\int_{-\infty}^{+\infty} f(x)dx$.

LEMMA C.1. *Let $f(x) \in \mathbb{C}^n$ be an H^1, complex vector-valued function, and $H(x) \in \mathbb{C}^{n\times n}$ a Hermitian, C^1 complex matrix-valued function. Then,*

$$\Re e\langle f, Hf'\rangle = -\Re e\langle f, (Hf)'\rangle = -(1/2)\langle f, H'f\rangle, \tag{C.1}$$

where "$\prime$" as usual denotes d/dx. Likewise, if $K(x) \in \mathbb{C}^{n\times n}$ is an anti-Hermitian C^1 complex matrix-valued function, then

$$\Im m\langle f, Kf'\rangle = -\Im m\langle f, (Kf)'\rangle = -(1/2)\langle f, K'f\rangle. \tag{C.2}$$

PROOF. The first equality in (C.1) follows upon integration by parts. Likewise, integrating by parts, we have

$$\begin{aligned}
\Re e\langle f, Hf'\rangle &= (1/2)(\langle f, Hf'\rangle + \langle Hf', f\rangle) \\
&= (1/2)(\langle f, Hf'\rangle + \langle f', Hf\rangle) \\
&= (1/2)(\langle f, Hf'\rangle - \langle f, Hf\rangle \\
&= -(1/2)\langle f, H'f\rangle,
\end{aligned}$$

verifying the second equality. By setting $H = -iK$ in (C.1), we obtain (C.2). □

Bibliography

[1] S. Benzoni-Gavage, D. Serre, and K. Zumbrun. Alternate Evans functions and viscous shock waves. *SIAM J. Math. Anal.*, 32(5):929–962 (electronic), 2001.

[2] L. Q. Brin. Numerical testing of the stability of viscous shock waves. *Math. Comp.*, 70(235):1071–1088, 2001.

[3] R. E. Caflisch and T.-P. Liu. Stability of shock waves for the Broadwell equations. *Comm. Math. Phys.*, 114(1):103–130, 1988.

[4] C. C. Conley and J. A. Smoller. Viscosity matrices for two-dimensional nonlinear hyperbolic systems. *Comm. Pure Appl. Math.*, 23:867–884, 1970.

[5] C. C. Conley and J. A. Smoller. Shock waves as limits of progressive wave solutions of higher order equations. *Comm. Pure Appl. Math.*, 24:459–472, 1971.

[6] N. Fenichel. Geometric singular perturbation theory for ordinary differential equations. *J. Differential Equations*, 31(1):53–98, 1979.

[7] K. O. Friedrichs. Symmetric hyperbolic linear differential equations. *Comm. Pure Appl. Math.*, 7:345–392, 1954.

[8] C. Fries. Nonlinear asymptotic stability of general small-amplitude viscous Laxian shock waves. *J. Differential Equations*, 146(1):185–202, 1998.

[9] C. Fries. Stability of viscous shock waves associated with non-convex modes. *Arch. Ration. Mech. Anal.*, 152(2):141–186, 2000.

[10] R. A. Gardner and K. Zumbrun. The gap lemma and geometric criteria for instability of viscous shock profiles. *Comm. Pure Appl. Math.*, 51(7):797–855, 1998.

[11] I. Gasser and P. Szmolyan. A geometric singular perturbation analysis of detonation and deflagration waves. *SIAM J. Math. Anal.*, 24(4):968–986, 1993.

[12] D. Gilbarg. The existence and limit behavior of the one-dimensional shock layer. *Amer. J. Math.*, 73:256–274, 1951.

[13] P. Godillon. Linear stability of shock profiles for systems of conservation laws with semi-linear relaxation. *Phys. D*, 148(3-4):289–316, 2001.

[14] J. Goodman. Nonlinear asymptotic stability of viscous shock profiles for conservation laws. *Arch. Rational Mech. Anal.*, 95(4):325–344, 1986.

[15] J. Goodman. Remarks on the stability of viscous shock waves. In *Viscous profiles and numerical methods for shock waves (Raleigh, NC, 1990)*, pages 66–72. SIAM, Philadelphia, PA, 1991.

[16] D. Henry. *Geometric theory of semilinear parabolic equations*. Springer-Verlag, Berlin, 1981.

[17] D. Hoff. Global well-posedness of the Cauchy problem for the Navier-Stokes equations of non-isentropic flow with discontinuous initial data. *J. Differential Equations*, 95(1):33–74, 1992.

[18] D. Hoff and K. Zumbrun. Multi-dimensional diffusion waves for the Navier-Stokes equations of compressible flow. *Indiana Univ. Math. J.*, 44(2):603–676, 1995.

[19] D. Hoff and K. Zumbrun. Pointwise decay estimates for multidimensional Navier-Stokes diffusion waves. *Z. Angew. Math. Phys.*, 48(4):597–614, 1997.

[20] P. Howard and K. Zumbrun. Pointwise estimates and stability for dispersive-diffusive shock waves. *Arch. Ration. Mech. Anal.*, 155(2):85–169, 2000.

[21] J. Humpherys. Stability of Jin-Xin relaxation shocks. To appear., 2000.

[22] J. Humpherys. On the shock wave spectrum for isentropic gas dynamics with capillarity. Preprint, 2002.

[23] J. Humpherys and K. Zumbrun. Spectral stability of small-amplitude shock profiles for dissipative symmetric hyperbolic-parabolic systems. *Z. Angew. Math. Phys.*, 53(1):20–34, 2002.

[24] S. Jin and Z. P. Xin. The relaxation schemes for systems of conservation laws in arbitrary space dimensions. *Comm. Pure Appl. Math.*, 48(3):235–276, 1995.

[25] C. K. R. T. Jones. Geometric singular perturbation theory. In *Dynamical systems (Montecatini Terme, 1994)*, pages 44–118. Springer, Berlin, 1995.

[26] J. I. Kanel′. A model system of equations for the one-dimensional motion of a gas. *Differencial′nye Uravnenija*, 4:721–734, 1968.

[27] S. Kawashima. *Systems of a hyperbolic–parabolic composite type, with applications to the equations of magnetohydrodynamics*. PhD thesis, Kyoto University, 1983.

[28] S. Kawashima and A. Matsumura. Asymptotic stability of traveling wave solutions of systems for one-dimensional gas motion. *Comm. Math. Phys.*, 101(1):97–127, 1985.

[29] S. Kawashima, A. Matsumura, and K. Nishihara. Asymptotic behavior of solutions for the equations of a viscous heat-conductive gas. *Proc. Japan Acad. Ser. A Math. Sci.*, 62(7):249–252, 1986.

[30] S. Kawashima and Y. Shizuta. On the normal form of the symmetric hyperbolic-parabolic systems associated with the conservation laws. *Tohoku Math. J. (2)*, 40(3):449–464, 1988.

[31] M. Khodja. *Nonlinear stability of oscillatory traveling waves for some systems of hyperbolic conservation laws*. PhD thesis, University of Michigan, 1989.

[32] P. D. Lax. *Hyperbolic systems of conservation laws and the mathematical theory of shock waves*. Society for Industrial and Applied Mathematics, Philadelphia, Pa., 1973. Conference Board of the Mathematical Sciences Regional Conference Series in Applied Mathematics, No. 11.

[33] H. Liu. Asymptotic stability of relaxation shock profiles for hyperbolic conservation laws. Preprint, 2000.

[34] T.-P. Liu. Nonlinear stability of shock waves for viscous conservation laws. *Mem. Amer. Math. Soc.*, 56(328):v+108, 1985.

[35] T.-P. Liu. Pointwise convergence to shock waves for viscous conservation laws. *Comm. Pure Appl. Math.*, 50(11):1113–1182, 1997.

[36] T.-P. Liu and Y. Zeng. Large time behavior of solutions for general quasilinear hyperbolic-parabolic systems of conservation laws. *Mem. Amer. Math. Soc.*, 125(599):viii+120, 1997.

[37] A. Majda and R. L. Pego. Stable viscosity matrices for systems of conservation laws. *J. Differential Equations*, 56(2):229–262, 1985.

[38] C. Mascia and K. Zumbrun. Pointwise Green's bounds and stability of relaxation shocks. Preprint, 2000.

[39] C. Mascia and K. Zumbrun. Stability of viscous shock profiles for dissipative symmetric hyperbolic-parabolic systems. Preprint, 2000.

[40] A. Matsumura and T. Nishida. Initial-boundary value problems for the equations of motion of compressible viscous and heat-conductive fluids. *Comm. Math. Phys.*, 89(4):445–464, 1983.

[41] A. Matsumura and K. Nishihara. On the stability of travelling wave solutions of a one-dimensional model system for compressible viscous gas. *Japan J. Appl. Math.*, 2(1):17–25, 1985.

[42] R. L. Pego. Stable viscosities and shock profiles for systems of conservation laws. *Trans. Amer. Math. Soc.*, 282(2):749–763, 1984.

[43] R. Plaza and K. Zumbrun. An Evans function approach to stability of small amplitude stability. Preprint, 2002.

[44] D. H. Sattinger. On the stability of waves of nonlinear parabolic systems. *Advances in Math.*, 22(3):312–355, 1976.

[45] Y. Shizuta and S. Kawashima. Systems of equations of hyperbolic-parabolic type with applications to the discrete Boltzmann equation. *Hokkaido Math. J.*, 14(2):249–275, 1985.

[46] M. Slemrod. A limiting "viscosity" approach to the Riemann problem for materials exhibiting change of phase. *Arch. Rational Mech. Anal.*, 105(4):327–365, 1989.

[47] J. Smoller. *Shock waves and reaction-diffusion equations.* Springer-Verlag, New York, second edition, 1994.

[48] J. A. Smoller and C. C. Conley. Shock waves as limits of progressive wave solutions of higher order equations. II. *Comm. Pure Appl. Math.*, 25:133–146, 1972.

[49] A. Szepessy and Z. P. Xin. Nonlinear stability of viscous shock waves. *Arch. Rational Mech. Anal.*, 122(1):53–103, 1993.

[50] H. Weyl. Shock waves in arbitrary fluids. *Comm. Pure Appl. Math.*, 2:103–122, 1949.

[51] Y. Zeng. L^1 asymptotic behavior of compressible, isentropic, viscous 1-D flow. *Comm. Pure Appl. Math.*, 47(8):1053–1082, 1994.

[52] K. Zumbrun. Stability of viscous shock waves. Lecture Notes, Indiana University, 1998.

[53] K. Zumbrun. Multidimensional stability of planar viscous shock waves. Lecture notes, TMR summer school, 1999.

[54] K. Zumbrun. Dynamical stability of phase transitions in the p-system with viscosity-capillarity. *SIAM J. Appl. Math.*, 60(6):1913–1924 (electronic), 2000.

[55] K. Zumbrun and P. Howard. Pointwise semigroup methods and stability of viscous shock waves. *Indiana Univ. Math. J.*, 47(3):741–871, 1998.

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