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ADMISSIBILITY OF VISCOUS-DISPERSIVE SYSTEMS

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Abstract. We consider the class of viscous-dispersive and higher-order conservation laws. We generalize the work of Kawashima and Shizuta, and others, by extending to higher-order the notions of symmetrizability, strict dissipativity, and genuine coupling. We prove, for symmetrizable systems, that strict dissipativity is equivalent to both (i) genuine coupling and (ii) the existence of a skew-symmetric compensating function.

Keywords: Viscous-dispersive systems; strict dissipativity; genuine coupling; compensating function; traveling wave stability.

1. Introduction

In recent years, there have been numerous advances in the stability theory of front propagation for systems of viscous conservation laws

$$u_t + f(u)_x = (B(u)u_x)_x. \tag{1.1}$$

In particular, for a very general class of hyperbolic-parabolic systems with degenerate viscosities, Zumbrun and collaborators [27, 13], generalizing the earlier work of [6, 40, 20, 36, 22, 29, 17] and others, have recently proven nonlinear stability for small-amplitude fronts and have also made considerable progress in the large-amplitude case [25, 26]. Their work relies heavily on the use of a skew-symmetric compensating function developed by Kawashima in his doctoral thesis and subsequent work with Shizuta [15, 16, 33, 18] (see also [23]).

Specifically, Kawashima and Shizuta showed that if systems of (1.1) are:

(i) symmetrizable, that is, there exists a symmetric positive-definite $A_0(u)$ (hereafter denoted $A_0(u) > 0$) so that $A_0(u)df(u)$ symmetric and $A_0(u)B(u)$ is symmetric and positive semi-definite (hereafter denoted $A_0(u)B(u) \geq 0$), and
(ii) genuinely coupled, that is, no eigenvector of $df(u)$ is in the kernel of $B(u)$,

then there exists a skew-symmetric compensating function $K(u)$ satisfying

$$A_0(u)B(u) + [K(u), A_0(u)\, df(u)] \geq \theta(u)I > 0, \tag{1.2}$$

for some scalar function $\theta(u) > 0$, or in other words, the left-hand side of (1.2) is positive-definite. Note that $A_0(u)B(u)$ is only positive semi-definite, and it is precisely the addition of the commutator term in (1.2) that is the key to strict definiteness, thus yielding the coercive expression needed in the energy estimates used to prove stability (see, for example, [13, 27]).

We remark that the compensating function methods of Kawashima and Shizuta also apply to relaxation models (see, for example, [39, 37])

$$u_t + f(u)_x = Q(u), \tag{1.3}$$

with similarly defined properties of symmetrizability and genuine coupling. Zumbrun and collaborators [24, 32], generalizing the work of [21, 12, 19, 1, 3] and others, have also proven stability for small-amplitude fronts for these systems and have likewise made advances toward understanding the large-amplitude case.

In this paper, we generalize the work of Shizuta and Kawashima [33] by considering systems of conservation laws with added relaxation, viscosity, dispersion, and/or other higher-order effects

$$u_t + f(u)_x = Q(u) + (B(u)u_x)_x + (C(u)u_{xx})_x + \cdots. \tag{1.4}$$

We likewise prove the existence of a skew-symmetric compensating function for genuinely coupled symmetrizable systems, yielding a similar inequality to (1.2). We then use this inequality to prove strict dissipativity — a necessary condition for traveling wave stability.

This paper is organized as follows: in Sec. 2, we review Kawashima's theorem, and state, for higher-order systems, the definitions of strict dissipativity and genuine coupling. In Sec. 3, we state the main result of this paper, Theorem 3.3, which equates for symmetric systems, the properties of strict dissipativity, genuine coupling, and the existence of a skew-symmetric compensating function K described above. In Sec. 4, we prove a series of lemmas from linear algebra that are used in the proofs of Theorem 3.3. These lemmas are generalizations of those developed by Ellis and Pinsky [4] and Shizuta and Kawashima [33] and are of independent interest. In Sec. 5, we prove Theorem 3.3. This proof generalizes the results of [33, 15, 16, 18] to higher-order systems of the form (1.4). The most challenging part of the proof is to show that the existence of a skew-symmetric compensating function satisfying an inequality that is analogous to (1.2) implies strict dissipativity. Our approach is similar to that of [33] in that it combines a Friedrichs estimate with a Kawashima-type estimate. We remark that this could also be proved via Lyapunov's stability theorem, however, the approach given herein motivates energy estimates used in the stability analysis of traveling waves.

It is important to note that Theorem 3.3 is only stated and proven for symmetric systems. In Sec. 6, we define symmetrizability for higher-order systems. We then restate Theorem 3.3 in its full generality to accommodate symmetrizable systems. As a working example, we apply our analysis to Slemrod's model of an isentropic gas with capillarity. We remark that this model is not symmetrizable in the traditional sense, but by extending our notion of a symmetrizer from a positive-definite matrix to a positive-definite differential operator, it can be symmetrized and thus the results of this work can be applied. Interestingly, with our generalized definition of a symmetrizer, there are profound consequences as to what it means to have a convex entropy for higher-order systems. In Sec. 7, we discuss this point and others to conclude the paper.

2. Background

Kawashima's theory develops by considering the second-order constant coefficient systems

$$v_t = Lv := -Av_x + Bv_{xx}, \quad x \in \mathbb{R}, \ t > 0, \ v \in \mathbb{R}^m, \tag{2.1}$$

where A and B are symmetric and $B \geq 0$. By taking the Fourier transform, the evolution of (2.1) reduces to solving the eigenvalue problem

$$(\lambda + i\xi A + \xi^2 B)\hat{v} = 0. \tag{2.2}$$

We have the theorem:

Theorem 2.1 (Shizuta−Kawashima [33]). *The following statements are equivalent*:

(i) *L is strictly dissipative*, *that is*, $\Re e(\lambda(\xi)) < 0$ *for all* $\xi \neq 0$.
(ii) *L is genuinely coupled*, *that is*, *no eigenvalue of A is in* $\mathcal{N}(B)$.
(iii) *There exists a skew-Hermitian K such that* $[K, A] + B > 0$.

In this paper, we generalize this theorem by considering the general linear system

$$v_t = \mathcal{L}v := -\sum_{k=0}^{n} D_k \partial_x^k v, \quad x \in \mathbb{R}, \ t > 0, \ v \in \mathbb{R}^m, \tag{2.3}$$

where each $m \times m$ matrix D_k is constant. Likewise, by taking the Fourier transform, the evolution of (2.3) reduces to solving the eigenvalue problem

$$\lambda\hat{v} + \sum_{k=0}^{n} (i\xi)^k D_k \hat{v} = 0. \tag{2.4}$$

We simplify by separating out odd- and even-ordered terms in (2.4) to get

$$(\lambda + i\xi A(\xi) + B(\xi))\hat{v} = 0 \tag{2.5}$$

where

$$A(\xi) := \sum_{k \text{ odd}} D_k (i\xi)^{k-1} \quad \text{and} \quad B(\xi) := \sum_{k \text{ even}} (-1)^{k/2} D_k \xi^k. \tag{2.6}$$

We refer to the matrix-valued terms $A(\xi)$ and $B(\xi)$, respectively, as the generalized flux and generalized viscosity. We have the following definitions:

Definition 2.2.

(i) $\mathcal{L}$ is called strictly dissipative if for each $\xi \neq 0$, we have that $\Re e(\lambda(\xi)) < 0$.
(ii) $\mathcal{L}$ is said to be genuinely coupled if no eigenvector of $A(\xi)$ is in $\mathcal{N}(B(\xi))$, for all fixed $\xi \neq 0$.

We remark that genuine coupling is physically relevant. For example, in the case of viscous or relaxed conservation laws, it has been shown in many cases that genuine coupling implies time-asymptotic smoothing (see [28, 23, 15, 16, 8, 9, 10, 38]). We remark that a loss of coupling in these instances means that a purely hyperbolic direction exists whereby discontinuous "shock wave" solutions can persist. Mathematically, it is easy to see that genuine coupling is a necessary condition for strict dissipativity. The main result of this paper tells us that for either symmetric or symmetrizable systems, it is also a sufficient condition.

3. Main Result

In this section, we state the main result of the paper. Indeed, we show that for symmetric systems, the properties of strict dissipativity, genuine coupling, and the existence of a skew-symmetric compensating function K are all equivalent. We make the following assumptions:

(H1) $A(\xi)$ is symmetric and of constant multiplicity in ξ.
(H2) $B(\xi) \geq 0$ (symmetric and positive semi-definite).

Lemma 3.1. *Assuming* (H1), *the distinct eigenvalues* $\{\mu_j(\xi)\}_{j=1}^r$ *of* $A(\xi)$ *and their corresponding orthogonal eigenprojections* $\{\pi_j(\xi)\}_{j=1}^r$ *are real-analytic in* ξ.

Proof. This is a direct consequence of [14, Theorem II.6.1, p. 120]. □

Lemma 3.2. *Assuming* (H2), *we have that* $\mathcal{L}$ *is genuinely coupled iff*

$$\theta(\xi) = \inf_{\|x\|=1} \sum_{j=1}^{r} \langle \pi_j(\xi)x, B(\xi)\pi_j(\xi)x\rangle. \tag{3.1}$$

is positive for all $\xi \neq 0$.

Proof. Assume L is genuinely coupled and suppose that $\theta(\xi) = 0$. Then each $\langle P_i(\xi)x, B(\xi)P_i(\xi)x\rangle = 0$ for some unit vector x. This implies that each $P_i(\xi)x = 0$, and thus $x = \sum P_i(\xi)x = 0$, which is a contradiction since $\|x\| = 1$. Conversely assume that $\theta(\xi) > 0$ and suppose L is not genuinely coupled for some fixed $\xi \neq 0$, then there exists a unit vector x such that $P_j(\xi)x = x$ and $Bx = 0$. However this is a contradiction since $\langle P(\xi)x, B(\xi)P_j(\xi)x\rangle = 0$. □

Theorem 3.3. *Given* (H1) *and* (H2) *above, the following statements are equivalent*:

(i) *$\mathcal{L}$ is strictly dissipative.*
(ii) *$\mathcal{L}$ is genuinely coupled.*
(iii) *There exists a real-analytic skew-Hermitian matrix-valued $K(\xi)$ such that $[K(\xi), A(\xi)] + B(\xi) > 0$ for all $\xi \neq 0$.*

The proof is given in Sec. 5.

4. Lemmata from Linear Algebra

In this section, we state and prove four lemmata used in our analysis. We generalize the work of Ellis and Pinsky [4] and Shizuta and Kawashima [33] (see also [11, 23]) by developing, in the language of eigenprojections, the spectral decomposition of the commutator operator $[A, \cdot]$. This allows for the compensating function K, developed by Kawashima, to be expressed in closed form as a Drazin inverse [2] of the commutator operator. By considering symmetric real-analytically varying matrices $A(\xi)$, we likewise obtain the corresponding real-analytically varying compensating function $K(\xi)$, which is the key to extending Kawashima's program to arbitrarily higher-order symmetrizable systems.

Let M_n denote the set of $n \times n$ matrices over $\mathbb{C}$ with the Frobenius inner product

$$\langle X, Y \rangle = \operatorname{tr}(X^* Y).$$

Given $A \in M_n$, we define the commutator or *Ad* operator on M_n as

$$\operatorname{Ad}_A(X) = [A, X].$$

The following basic properties are straightforward to prove:

Lemma 4.1.

(i) *Linearity*:
$$Ad_A(aX + bY) = a(Ad_A X) + b(Ad_A Y).$$
(ii) *Leibniz Rule*:
$$Ad_A(BC) = (Ad_A B)C + B(Ad_A C).$$
(iii) *Jacobi Identity*:
$$Ad_{[A,B]} = Ad_A Ad_B - Ad_B Ad_A.$$
(iv) *Adjoint preservation*:
$$(Ad_A)^* = Ad_{A^*}.$$

It follows from (iii) [respectively, (iv)] above that Ad_A is normal [respectively, Hermitian] if and only if A is normal [respectively, Hermitian]. The following lemma describes the spectral decomposition of the commutator operator:

Lemma 4.2. *Let A be semi-simple. Denote the distinct eigenvalues of A as $\{\mu_j\}_{j=1}^r$ with corresponding eigenprojections $\{\pi_j\}_{j=1}^r$. Then Ad_A is also semi-simple.*

The eigenvalues of Ad_A *are of the form* $\mu_i - \mu_j$ *and have the corresponding eigenprojections*

$$\pi_{ij}(X) := \pi_i X \pi_j. \tag{4.7}$$

Moreover, the following hold

(i) *Idempotence and independence*:

$$\pi_{ij}(\pi_{mn}(X)) = \delta_{im}\delta_{jn}\pi_{ij}(X). \tag{4.8}$$

(ii) *A-invariance*:

$$\mathrm{Ad}_A(\pi_{ij}(X)) = \pi_{ij}(\mathrm{Ad}_A(X)) = (\mu_i - \mu_j)\pi_{ij}(X). \tag{4.9}$$

(iii) *Completeness*:

$$\sum_{i,j} \pi_{ij}(X) = X. \tag{4.10}$$

Proof.

(i)

$$\pi_{ij}(\pi_{mn}(X)) = \pi_i(\pi_m X \pi_n)\pi_j = \delta_{im}\delta_{jn}\pi_i X \pi_j = \delta_{im}\delta_{jn}\pi_{ij}(X).$$

(ii)

$$\begin{aligned}\mathrm{Ad}_A(\pi_{ij}(X)) &= A\pi_i X \pi_j - \pi_i X \pi_j A = (\mu_i - \mu_j)\pi_{ij}(X),\\ \pi_{ij}(\mathrm{Ad}_A(X)) &= \pi_i A X \pi_j - \pi_i X A \pi_j = (\mu_i - \mu_j)\pi_{ij}(X).\end{aligned}$$

(iii)

$$\sum_{i,j=1}^{r} \pi_{ij}(X) = \sum_{i,j=1}^{r} \pi_i X \pi_j = \left(\sum_{i=1}^{r} \pi_i\right) X \left(\sum_{j=1}^{r} \pi_j\right) = X. \qquad \square$$

The next lemma shows how to solve the inverse commutator problem using the Drazin inverse [2] or reduced resolvent.

Lemma 4.3. *Let* A *be semi-simple. Define following linear operator on* M_n:

$$\Pi_A(X) := \sum_{i=j} \pi_{ij}(X) = \sum_{j=1}^{r} \pi_j X \pi_j. \tag{4.13}$$

The following hold:

(i) Π_A *is the projection onto* $\mathcal{N}(\mathrm{Ad}_A)$ *along* $\mathcal{R}(\mathrm{Ad}_A)$.

(ii) *For each* $B \in M_n$ *there exists* $K \in M_n$ *such that*

$$B = \Pi_A(B) + \mathrm{Ad}_A(K). \tag{4.14}$$

Moreover the canonical solution K, *which we call the compensating matrix, is of the form*

$$K = \sum_{i \neq j} \frac{\pi_{ij}(B)}{\mu_i - \mu_j}. \tag{4.15}$$

This is the Drazin inverse or reduced resolvent of the commutator operator.

Proof.

(i) Π_A is clearly a projection since

$$\Pi_A(\Pi_A(X)) = \sum_{j,k} \pi_j \pi_k X \pi_k \pi_j = \sum_j \pi_j X \pi_j = \Pi_A(X).$$

Since $\mathrm{Ad}_A(\pi_{jj}(X)) = \pi_{jj}(\mathrm{Ad}_A(X)) = 0$, from (4.9), we have that both $\mathcal{R}(\Pi_A) \subset \mathcal{N}(\mathrm{Ad}_A)$ and $\mathcal{R}(\mathrm{Ad}_A) \subset \mathcal{N}(\Pi_A)$. To show that $\mathcal{N}(\mathrm{Ad}_A) \subset \mathcal{R}(\Pi_A)$, we suppose $[A, X] = 0$. Then each $[\pi_j, X] = 0$ and hence

$$\Pi_A(X) = \sum_j \pi_j X \pi_j = \sum_j X \pi_j \pi_j = X \sum_j \pi_j = X.$$

Finally we show that $\mathcal{N}(\Pi_A) \subset \mathcal{R}(\mathrm{Ad}_A)$. If $\Pi_A(X) = 0$, set

$$K = \sum_{i \neq j} \frac{\pi_{ij}(X)}{\mu_i - \mu_j}.$$

Then $\mathrm{Ad}_A(K) = X$, which implies that $X \in \mathcal{R}(\mathrm{Ad}_A)$.

(ii) Since $I - \Pi_A$ is the projection onto $\mathcal{R}(\mathrm{Ad}_A)$ along $\mathcal{N}(\mathrm{Ad}_A)$, it follows that $B - \Pi_A(B) \in \mathcal{R}(\mathrm{Ad}_A)$. Thus, there is some $K \in M_n$ such that $B - \Pi_A(B) = \mathrm{Ad}_A(K)$. The explicit form is obtained by the Drazin inverse (or reduced resolvent) of Ad_A and has the form of (4.15). Indeed, substituting (4.15) into (4.14) yields

$$\begin{aligned}\Pi_A(B) + \mathrm{Ad}_A(K) &= \sum_{i=j} \pi_{ij}(B) + \sum_{i \neq j} \frac{\mathrm{Ad}_A(\pi_{ij}(B))}{\mu_i - \mu_j} \\ &= \sum_{i=j} \pi_{ij}(B) + \sum_{i \neq j} \pi_{ij}(B) \\ &= B.\end{aligned}$$

□

The final lemma shows how the theory simplifies when A and B are Hermitian and $B \geq 0$. We have:

Lemma 4.4. *Assume A and B are Hermitian and $B \geq 0$. Then*:

(i) $\Pi_A(B)$ *is Hermitian.*
(ii) $\Pi_A(B) \geq 0$.
(iii) *The canonical solution K given in* (4.15) *is skew-Hermitian.*

Proof.

(i) Since A is Hermitian (and is thus normal) its eigenprojections are orthogonal, that is, $\pi_j^* = \pi_j$. Hence

$$\Pi_A(B)^* = \left(\sum_j \pi_j B \pi_j \right)^* = \sum_j \pi_j B \pi_j = \Pi_A(B).$$

(ii) Let $x_j = \pi_j x$. Then each $\langle x_j, Bx_j\rangle \geq 0$ since $B \geq 0$. Hence,

$$\langle x, \Pi_A(B)x\rangle = \sum_j \langle x, \pi_j B\pi_j x\rangle = \sum_j \langle \pi_j x, B\pi_j x\rangle = \sum_j \langle x_j, Bx_j\rangle \geq 0.$$

(iii) Note that $\pi_{ij}(B)^* = \pi_{ji}(B)$. Hence,

$$K^* = \sum_{i\neq j} \frac{\pi_{ij}(B)^*}{\mu_i - \mu_j} = \sum_{i\neq j} \frac{\pi_{ji}(B)}{\mu_i - \mu_j} = -\sum_{i\neq j} \frac{\pi_{ji}(B)}{\mu_j - \mu_i} = -K. \qquad \square$$

5. Proof of Theorem 3.3

The proof goes as follows:

(i) $\Rightarrow$ (ii): Suppose for some fixed $\xi \neq 0$ that both $A(\xi)v = \mu(\xi)v$ and $v \in \mathcal{N}(B(\xi))$. Then

$$(i\xi A(\xi) + B(\xi))v = i\xi\mu(\xi)v.$$

Hence, $\Re e\lambda(\xi) = \Re e(-i\xi\mu(\xi)) = 0$, which contradicts (i).

(ii) $\Rightarrow$ (iii): Let $\xi \neq 0$ be fixed. By Lemmas 4.3 and 4.4, there exists a skew-Hermitian $K(\xi)$ satisfying

$$B(\xi) = \Pi_{A(\xi)}(B(\xi)) + \mathrm{Ad}_{A(\xi)}(K(\xi)).$$

In addition, we know that $\Pi_{A(\xi)}(B(\xi))$ is Hermitian and positive semi-definite. Note that $\Pi_{A(\xi)}(B(\xi)) \geq \theta(\xi)I$. Since genuine coupling implies that $\theta(\xi) > 0$, strict positive-definiteness directly follows. To show that $K(\xi)$ is real-analytic, we need only apply Lemma 3.1 to the canonical choice of $K(\xi)$ given in (4.15). Note that

$$K(\xi) = \sum_{i\neq j} \frac{\pi_i(\xi)B(\xi)\pi_j(\xi)}{\mu_i(\xi) - \mu_j(\xi)}$$

is well-defined for all $\xi \neq 0$ and real-analytic in ξ. Note that constant multiplicity is used here to keep the denominator of $K(\xi)$ bounded away from zero.

(iii) $\Rightarrow$ (i): We suppose that for some fixed ξ there is a λ and v satisfying (2.5). We combine two spectral energy estimates. First, by taking the inner product of (2.5) with v and taking the real part, we get the following standard Friedrichs-type estimate [5]:

$$\Re e(\lambda)\|v\|^2 + \langle v, B(\xi)v\rangle = 0. \tag{5.4}$$

We also have

$$\Re e(\lambda)\|v\|^2 + \frac{1}{\|B(\xi)\|}\|B(\xi)v\|^2 \leq 0. \tag{5.5}$$

Note that both (5.4) and (5.5) imply that $\Re e(\lambda) \leq 0$. For the second estimate, we multiply (2.5) by the Hermitian matrix $2i\xi K(\xi)$ and take the inner product with v to get

$$2\xi^2\langle v, K(\xi)A(\xi)v\rangle = 2i\lambda\xi\langle v, K(\xi)v\rangle + 2i\xi\langle v, K(\xi)B(\xi)v\rangle \tag{5.6}$$

Taking the real part and using Young's inequality gives

$$\begin{aligned}
&2\xi^2\langle v, [K(\xi), A(\xi)]v\rangle \\
&\quad = 4i\xi\Re e(\lambda)\langle v, K(\xi)v\rangle + 2i\xi\langle v, (K(\xi)B(\xi) + B(\xi)K(\xi))v\rangle \\
&\quad \leq 4|\Re e(\lambda)||\xi|\|K(\xi)\|\|v\|^2 + 4|\xi|\|v\|\|K(\xi)\|\|B(\xi)v\| \\
&\quad \leq 4|\Re e(\lambda)||\xi|\|K(\xi)\|\|v\|^2 + \theta(\xi)\xi^2\|v\|^2 + 4\frac{\|K(\xi)\|^2}{\theta(\xi)}\|B(\xi)v\|^2,
\end{aligned}$$

where $[K(\xi), A(\xi)] + B(\xi) \geq \theta(\xi)I > 0$. Multiplying $2\xi^2$ by (5.4) and adding gives

$$\xi^2\theta(\xi)\|v\|^2 + 4\Re e(\lambda)|\xi|\|K(\xi)\|\|v\|^2 \leq 4\frac{\|K(\xi)\|^2}{\theta(\xi)}\|B(\xi)v\|^2.$$

Using (5.5) to cancel the right-hand side yields

$$\xi^2\theta(\xi)\|v\|^2 + 4\Re e(\lambda)|\xi|\|K(\xi)\|\|v\|^2 + 4\Re e(\lambda)\frac{\|K(\xi)\|^2}{\theta(\xi)}\|B(\xi)\|\|v\|^2 \leq 0.$$

Hence

$$\Re e(\lambda) \leq \frac{-\xi^2\theta(\xi)^2}{4|\xi|\|K(\xi)\|\theta(\xi) + 4\|K(\xi)\|^2\|B(\xi)\|}. \tag{5.9}$$

Thus $\Re e(\lambda(\xi)) \leq 0$ for all $\xi \neq 0$. This completes the proof.

6. Symmetrizability

In both viscous and relaxed conservation laws, symmetrizability has been proven, repeatedly, to be important (see, for example, [15, 23]). Pego was the first to consider symmetrizability for the viscous-dispersive and higher order conservation laws [30, 31]. Indeed he proved that term-wise symmetrizability (defined below) was a sufficient condition for the admissibility of higher-order systems. We have the following:

Theorem 6.1 (Pego [30, 31]). *The system in* (2.3) *is strictly dissipative if it is term-wise symmetrizable*, *that is*, *there exists a constant matrix* $A_0 > 0$ *such that*:

(i) A_0D_j *is symmetric if* j *is odd.*
(ii) $(i\xi)^jA_0D_j$ *is negative-definite if* j *is even.*

We remark, however, that Slemrod's model (given below) is not term-wise symmetrizable, but by extending our notion of a symmetrizer from a positive-definite matrix to a positive-definite differential operator, we are able to symmetrize it.

Consider Slemrod's model [34, 35, 7] for a compressible isentropic gas with capillarity, defined as

$$\begin{aligned}
v_t - u_x &= 0, \\
u_t + p(v)_x &= (b(v)u_x)_x + dv_{xxx},
\end{aligned} \tag{6.1}$$

where physically, v is the specific volume, u is the velocity in Lagrangian coordinates, $p(v)$ is the pressure law for an ideal gas, that is, $p'(v) < 0$ and $p''(v) > 0$, $b(v)$ is the viscosity, satisfying $b(v) > 0$, $b'(v) \leq 0$, and the capillarity term $d < 0$ is constant. The Fourier-transformed constant coefficient case is given as

$$\begin{aligned} v_t - i\xi u &= 0, \\ u_t - i\xi(c^2 - d\xi^2)v + \xi^2 bu &= 0, \end{aligned} \tag{6.2}$$

where both $p'(v) = -c^2$ and $b > 0$ are constant. Note that (6.2) can only be symmetrized by permitting a ξ-valued symmetrizer, namely,

$$A_0(\xi) = \begin{pmatrix} c^2 - \xi^2 d & 0 \\ 0 & 1 \end{pmatrix}, \tag{6.3}$$

which is symmetric and positive-definite, as required. Left multiplying (6.2) by $A_0(\xi)$ yields

$$\lambda A_0(\xi)\begin{pmatrix} v \\ u \end{pmatrix} + i\xi\begin{pmatrix} 0 & -c^2 + \xi^2 d \\ -c^2 + \xi^2 d & 0 \end{pmatrix}\begin{pmatrix} v \\ u \end{pmatrix} = \xi^2\begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix}\begin{pmatrix} v \\ u \end{pmatrix},$$

which is symmetrized, and thus satisfies (H1) and (H2) above. Hence, we propose the following definition for $\mathcal{L}$ in (2.3).

Definition 6.2. $\mathcal{L}$ is called symmetrizable if there exists a symmetric, real-analytic matrix-valued function $A_0(\xi) > 0$ such that both $A_0(\xi)A(\xi)$ and $A_0(\xi)B(\xi)$ are symmetric, and $A_0(\xi)B(\xi) \geq 0$.

With this more general notion of symmetrizability, we can easily extend Theorem 3.3 to the following:

Theorem 6.3 (Symmetrizable version). *If $A_0(\xi)$ is a symmetrizer for $\mathcal{L}$, then the following statements are equivalent*:

(i) *$\mathcal{L}$ is strictly dissipative.*
(ii) *$\mathcal{L}$ is genuinely coupled.*
(iii) *There exists a real-analytic skew-Hermitian matrix-valued $K(\xi)$ such that $[K(\xi), A_0(\xi)A(\xi)] + A_0(\xi)B(\xi) > 0$ for all $\xi \neq 0$.*

7. Brief Discussion

We remark that extending the idea of a symmetrizer to a differential operator has profound consequences as to what it means to have a convex entropy for higher-order systems. As the existence of an entropy is both physically and mathematically important, further examination is warranted.

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