Math 316 Hwk 2

Problem 1. *Prove that* $|\|\mathbf{x}\| - \|\mathbf{y}\|| \leq \|\mathbf{x} - \mathbf{y}\|$.

Problem 2. *A function* $f : \mathbb{R}^m \longrightarrow \mathbb{R}^n$ *is said to be Lipschutz with constant* $L > 0$ *on the set* $S \subset \mathbb{R}^m$ *if for all* $\mathbf{x}, \mathbf{y} \in S$, *we have*

$$\|f(\mathbf{x}) - f(\mathbf{y})\| \leq L\|\mathbf{x} - \mathbf{y}\|.$$

Prove that if f *is Lipschutz on* S, *then* f *is uniformly continuous on* S.

Problem 3. *Let* $\{(X_k, d_k)\}_{k=1}^{\infty}$ *be a countably infinite collection of uniformly bounded metric spaces (meaning* $d_k(p, q) \leq M$ *for all* $k \in \mathbb{N}$*). Let* $X = \prod_{k=1}^{\infty} X_k$. *For* $\{x_k\}_{k=1}^{\infty}$ *and* $\{y_k\}_{k=1}^{\infty}$ *in* X, *define*

$$\rho(\{x_k\}_{k=1}^{\infty}, \{y_k\}_{k=1}^{\infty}) = \sum_{k=1}^{\infty} 2^{-k} d_k(x_k, y_k).$$

Show that (X, ρ) *is a metric space.*

Problem 4. *Prove that* $(\overline{E})^c = (E^c)^\circ$.

Problem 5. *Let* X *be an infinite set. Define*

$$d(p, q) = \begin{cases} 1 & p \neq q \\ 0 & p = q. \end{cases}$$

Prove that (X, d) *is a metrix space. Describe which sets are open, which are closed, and which are compact.*

Problem 6. *Given an example of a set with exactly* 3 *limit points.*

Problem 7. *Give an example of an open cover of* $(0, 1) \times (0, 1)$, *which has no finite subcover.*

Problem 8. *If* $\mathbf{b}_n > n$ *for each* n, *prove that* $\{\mathbf{b}_n\}_{n=1}^{\infty}$ *is unbounded.*

Problem 9. *Given two sets* C *and* D *in* $\mathbb{R}^n$, *define the distance* $d(C, D)$ *to be*

$$d(C, D) = \inf\{\|\mathbf{c} - \mathbf{d}\| \mid \mathbf{c} \in C, \mathbf{d} \in D\}.$$

Prove that:

(a). *For fixed* $\mathbf{d}$*, the function* $h(\mathbf{x}) = \|\mathbf{x} - \mathbf{d}\|$ *is continuous.*

(b). *For a closed set* D*,* $\exists\, \mathbf{d} \in D$ *such that* $d(\mathbf{x}, D) = \|\mathbf{x} - \mathbf{d}\|$*. HINT: Choose* r *so that* $B(\mathbf{x}, r) \cap D \neq \emptyset$*, note that* $B(\mathbf{x}, r) \cap D$ *is compact and consider* h *on* $B(\mathbf{x}, r) \cap D$*.*

Problem 10. *Prove that:*

(a). *The function* $f(\mathbf{x}) = d(\mathbf{x}, D)$ *is continuous.*

(b). *If* C *is compact and* D *is closed, then there exists* $\mathbf{c} \in C$ *and* $\mathbf{d} \in D$ *such that* $d(C, D) = \|\mathbf{c} - \mathbf{d}\|$*. HINT: Look at* f *on* C*.*