Math 316 Hwk 3

Problem 1. *If $U \subset \mathbb{R}^n$ is open and $K \subset U$ is compact, prove that there is a compact set D such that $K \subset D^\circ$ and $D \subset U$.*

Problem 2. *Recall that a homeomorphism is bijective, continuous, and has a continuous inverse. Prove that there is no homeomorphism from $\mathbb{R}^2$ onto $\mathbb{R}$. HINT: Remove a point from each and consider a connectedness argument.*

Problem 3. *If f, g are continuous on the Euclidian spaces $\mathbb{R}^m$ into $\mathbb{R}$, prove that $f + g$ and $f \cdot g$ are continuous.*

Problem 4. *A hyperplane is a set of the form $\{\mathbf{x} \in \mathbb{R}^m \mid \mathbf{a}^T\mathbf{x} = b\}$, where $\mathbf{a} \in \mathbb{R}^m$, $\mathbf{a} \neq \mathbf{0}$, and $b \in \mathbb{R}$. Prove that a hyperplane is convex.*

Problem 5. *What is the distance between two parallel hyperplanes. Use the Euclidian metric in $\mathbb{R}^n$.*

Problem 6. *A halfspace is a set of the form $\{\mathbf{x} \in \mathbb{R}^m \mid \mathbf{a}^T\mathbf{x} \leq b\}$, where $\mathbf{a} \in \mathbb{R}^m$, $\mathbf{a} \neq \mathbf{0}$, and $b \in \mathbb{R}$. Prove that a halfspace is convex.*

Problem 7. *Under what conditions does one halfspace contain another? Use the Euclidian metric in $\mathbb{R}^n$.*

Problem 8. *We know that $\mathbb{R}$ is complete. Use this to show that $\mathbb{R}^n$ is complete in the Euclidian norm.*

Problem 9. *Prove carefully the following: If a Cauchy sequence $\{\mathbf{x}_k\}_{k=1}^{\infty}$ has a cluster point $\mathbf{x} \in X$, then it converges to $\mathbf{x}$.*

Problem 10. *If $\{\mathbf{x}_k\}_{k=1}^{\infty}$ and $\{\mathbf{y}_k\}_{k=1}^{\infty}$ are Cauchy sequences in (X, d), then $d(\mathbf{x}_n, \mathbf{y}_n)$ converges. Note that X may not be complete.*