# Math 316 Hwk 4

**Problem 1.** *Prove the following: If $f : [a, b] \longrightarrow [a, b]$ is continuous, then there exists $x \in [a, b]$ such that $f(x) = x$.*

**Problem 2.** *A set $E \subset \mathbb{R}^n$ is said to be path connected if for any $\mathbf{x}, \mathbf{y} \in E$ there is a continuous map $\gamma : [0, 1] \longrightarrow \mathbb{R}^n$ such that $\gamma([0, 1]) \subset E$, $\gamma(0) = \mathbf{x}$ and $\gamma(1) = \mathbf{y}$. Prove that a path connected space is connected.*

**Problem 3.** *Let*

$$f(x, y) = \begin{cases} 0 & x = y = 0 \\ \dfrac{2x^2y}{x^4 + y^2} & \textit{otherwise} \end{cases}$$

*Define $\phi(t) = (t, at)$ and $\psi(t) = (t, t^2)$. Show that:*

(a). $\lim_{t\to 0} f(\phi(t)) = 0$.

(b). $\lim_{t\to 0} f(\psi(t)) = 1$.

*What does this mean? Explain your results.*

**Problem 4.** *A function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is called periodic, if there exists a number $T > 0$ such that $f(x + T) = f(x)$ for all $x \in \mathbb{R}$. Show that a continuous periodic function is bounded an uniformly continuous on $\mathbb{R}$.*

**Problem 5.** *Show that the function $f(x) = (x+1)^{-1}$ is uniformly continuous on the interval $(0, \infty)$, but not on the interval $(-1, 0)$.*

**Problem 6.** *Determine if the limit of $f(x, y)$ exists at $(0, 0)$ for*

(a). $f(x, y) = \dfrac{\sqrt{xy}}{x^2 + y^2}$.

(b). $f(x, y) = \dfrac{xy}{x^2 + y^2}$.

(c). $f(x, y) = \dfrac{xy}{\sqrt{x^2 + y^2}}$.

**Definition.** *Let* $f : \mathbb{R}^m \longrightarrow \mathbb{R}^n$*. We have the following sets:*

(a). *The domain,* $\mathcal{D}(f) = \{\mathbf{x} \in \mathbb{R}^m \mid f(x) \text{ exists}\}$.

(b). *The range,* $\mathcal{R}(f) = \{\mathbf{y} \in \mathbb{R}^n \mid f(\mathbf{x}) = \mathbf{y} \text{ for some } \mathbf{x} \in \mathcal{D}(f)\}$.

(c). *The graph,* $\mathcal{G}(f) = \{(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^m \times \mathbb{R}^n \mid \mathbf{x} \in \mathcal{D}(f) \text{ and } f(\mathbf{x}) = \mathbf{y}\}$.

(d). *The level set of* $f$ *at* $\mathbf{b}$, $f^{-1}(\mathbf{b}) = \{\mathbf{x} \in \mathbb{R}^m \mid f(\mathbf{x}) = \mathbf{b}\}$.

**Problem 7.** *Sketch the domain, range, graph, and level sets of the function* $f(x, y) = \dfrac{1}{xy}$.

**Problem 8.** *Sketch the following:*

(a). *The range and graph of* $f(t) = (\cos t, \sin t)$.

(b). *The range of* $f(r, \theta) = (r\cos\theta, r\sin\theta, r)$, $0 \leq r \leq 1$ *and* $0 \leq \theta \leq 2\pi$.

(c). *The level set of* $f(x, y, z) = (x + y + z, x^2 + y^2 + z^2)$ *at* $(1, 1)$.

(d). *The range of* $f(\theta, \phi) = (\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi)$, $-\frac{\pi}{2} \leq \phi \leq \frac{\pi}{2}$ *and* $0 \leq \theta \leq 2\pi$.

**Problem 9.** *Let* $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$*. Find:*

(a). *A function whose range is the graph of* $f$.

(b). *A function whose level set at* $\mathbf{0}$ *is the graph of* $f$.

**Problem 10.** *Find the following:*

(a). *A function* $g : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ *so that* $g(f(x, y)) = (x, y)$*, where the function* $f$ *is defined by* $f(x, y) = (3x + 2y, x - y)$.

(b). *For the function* $F : \mathbb{R}^4 \longrightarrow \mathbb{R}^2$ *defined by*

$$F(x_1, x_2, x_3, x_4) = (x_1 - x_2 + x_3 + x_4, 2x_1 - 3x_2 + x_3 + 4x_4),$$

*let* $S$ *be the level set of* $F$ *at* $(1, -1)$*. Find (i) a function whose range is* $S$ *and (ii) a function whose graph is* $S$.