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Math 290
Practice Final Exam Key

Note that the first 10 questions are true–false. Mark A for true, B for false. Questions 11 through 20 are multiple choice. Mark the correct answer on your bubble sheet. Answers to the last five questions should be written directly on the exam, and should be written neatly and correctly. Questions 1 to 20 are worth 2.5 points each, and questions 21 to 25 are worth 10 points each.

1. For any three sets A, B, C, $A - (B \cup C) = (A - B) \cup (A - C)$.

   False. Consider $A = \{1, 2, 6, 7\}$, $B = \{2, 3, 4, 7\}$, $C = \{4, 5, 6, 7\}$.

2. The statement $P \Rightarrow Q$ is logically equivalent to the statement $(\neg P) \vee Q$.

   True.

P	Q	$\neg P$	$\neg P \vee Q$	$P \implies Q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

Table 1. Truth table.

3. If $x, y \in \mathbb{R}$ then $|x + y| \geq |x| + |y|$.

   False. Consider $x = 1$, $y = -1$.

4. The negation of the statement "There exist irrational numbers a, b such that a^b is rational." is the statement "For all irrational numbers a, b, it is true that a^b is irrational."

   True. $\neg(\exists a, b \in (\mathbb{R} - \mathbb{Q}),\ a^b \in \mathbb{Q}) \equiv \forall a, b \in (\mathbb{R} - \mathbb{Q}),\ a^b \in (\mathbb{R} - \mathbb{Q})$.

5. Let R be an equivalence relation on a nonempty set A, and denote the equivalence class of $a \in A$ by $[a]$. For $a, b \in A$, we have aRb if and only if $[a] = [b]$.

   True. Recall that $[a] := \{x \in A : aRx\}$. Since R is an equivalence relation, aRb implies bRa, and vice versa.

6. Let A be a nonempty set, and suppose $f : A \to A$ is a surjective function. Then f is injective.

   False. Consider $f : \mathbb{R} \to \mathbb{R}$ given by the rule $f(x) = (x - 1)x(x + 1)$.

7. Every nonempty subset of the positive integers has a least element.

   True. You can use induction to show that any finite subset of the natural numbers has a least element. Now let $A \subseteq \mathbb{N}$ be nonempty. Since A is nonempty, there exists $n \in A$. Either n is the lease element of A, or $A \cap \{1, 2, 3, ..., n - 1\}$ is nonempty. But this last set has a least element which necessarily is the least element of A.

8. If S is an uncountable set and $T \subseteq S$ is uncountable, then $|T| = |S|$.

   False. Let T be the set of real numbers, and let S be the union of the set of real numbers and the power set of the set of real numbers.

9. The GCD of 1357 and 1633 is between 50 and 100.

False. The Euclidean algorithm shows that GCD(1357,1633) = 23.

$$\begin{aligned} 1633 &= 1357(1) + 276 \\ 1357 &= 276(4) + 253 \\ 276 &= 253(1) + 23 \\ 253 &= 23(11) + 0 \end{aligned} \tag{1}$$

10. Let $a, b, c \in \mathbb{Z}$, with $a \neq 0$. If $a|bc$ and $a \nmid b$, then $a|c$.

False. Consider $a = 6$, $b = 3$, and $c = 2$.

Multiple Choice Questions

11. Which of the following definitions is incorrect?

    (a) An equivalence relation is a relation on a nonempty set which is reflexive, symmetric and transitive.
    (b) A function $f : A \to B$ is injective if for every $a_1, a_2 \in A$, $f(a_1) \neq f(a_2)$ implies that $a_1 \neq a_2$.
    (c) Two sets A and B have the same cardinality if there is a bijection $f : A \to B$.
    (d) A partition of a set A is a collection of nonempty pairwise disjoint subsets of A whose union is A.
    (e) Two integers a and b are relatively prime if their greatest common divisor is 1.
    (f) None of the above. All of these definitions are correct.

(b) is not correct. As stated, (b) indicates that f is a function. To fix the definition, replace both $\neq$ signs with $=$ signs.

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12. Which of the following would be the best method for proving the statement

$$P \Rightarrow (Q \vee R).$$

    (a) Assume P and Q and prove that R is true.
    (b) Assume P and $\neg Q$ and prove that R is true.
    (c) Assume P and Q and prove that R is false.
    (d) Assume $\neg Q$ or $\neg R$ and prove $\neg P$.
    (e) Assume Q and R and prove that P is true.

Answer: (b). You can prove the result using the case that Q holds, and the case that $\neg Q$ holds. In the first case, $Q \vee R$ holds since Q is true. In the second case, you must show R is true.

13. Which of the following is the contrapositive of the statement

If every $x \in S$ is prime, then every $x \in S$ is odd.

(a) If there is an $x \in S$ which is prime, then there is an $x \in S$ that is odd.
(b) If there is an $x \in S$ that is odd, then there is an $x \in S$ that is prime.
(c) If there is an $x \in S$ that is even, then there is an $x \in S$ that is not prime.
(d) If every $x \in S$ is odd, then every $x \in S$ is prime.
(e) If every x in S is not prime, then every $x \in S$ is odd.
(f) If every $x \in S$ is odd, then there is an $x \in S$ that is prime.
(g) If every $x \notin S$ is even, then every $x \notin S$ is not prime.

The answer is (c). The contrapositive of $P \implies Q$ is $\neg Q \implies \neg P$.

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14. Which of the following statements is true.

(a) Let $x \in \mathbb{Z}$. If $4x + 7$ is odd then x is even.
(b) There exists a real number x such that $x^2 < x < x^3$.
(c) If $x \in \mathbb{Z}$ is odd, then $x^2 + x$ is even.
(d) Every odd integer is a sum of four odd integers.
(e) Let $x, y, z \in \mathbb{Z}$. If $z = x - y$ and z is even, then x and y are odd.
(f) For every two sets A and B, $(A \cup B) - B = A$.

The answer is (c). Note that $x^2 + x = x(x + 1)$. Either x is even, or $x + 1$ is even. You can use that to show that $x(x + 1)$ is even. To see that the other statements are false, consider: (a) $x = 1$, (b) $x^2 < x$ implies that $x > 0$ and $0 < x < 1$. But then $x^3 < x^2$. (d) The sum of any two odd integers is an even integer. (e) $x = 4$, $y = 2$.

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15. Which of the following is **not** an equivalence relation on the set $\mathbb{Z}$ of integers?

(a) xRy if $7|(x - y)$.
(b) xRy if $6|(x^2 - y^2)$
(c) xRy if $x + y \geq 0$.
(d) xRy if $x + y = 2x$.
(e) xRy if $x^2 - 2xy + y^2 \geq 0$.

The answer is (c). Consider $a = -2$, $b = 3$, and $c = -1$. Then aRb and bRc, but a does not relate to c.

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16. Evaluate the proposed proof of the following result. Choose the most complete correct answer.

Theorem: If two functions $f : A \to B$ and $g : B \to C$ are bijective, then $g \circ f$ is bijective.

Proof. Suppose that f and g are both bijective. Note that f and g are each both injective and surjective.

Let $c \in C$. Then, since g is surjective, $c = g(b)$ for some $b \in B$. Since f is surjective, $b = f(a)$ for some $a \in A$. Then $c = g(b) = g(f(a)) = (g \circ f)(a)$. Hence, $g \circ f$ is surjective.

Suppose that $a_1, a_2 \in A$, and $a_1 = a_2$. Then $f(a_1) = f(a_2)$, since f is injective, and $g(f(a_1)) = g(f(a_2))$ since g is injective. Hence $(g \circ f)(a_1) = (g \circ f)(a_2)$, so $g \circ f$ is injective.

Since $g \circ f$ is both injective and surjective, it is bijective. □

(a) The theorem and the proof are correct.
(b) The proof is correct but the theorem is false.
(c) The proof does not successfully prove that $g \circ f$ is surjective.

(d) The proof does not successfully prove that $g \circ f$ is injective.
(e) The proof proves neither that $g \circ f$ is surjective, nor that it is injective.
(f) The proof is irrelevant because injectivity and surjectivity have nothing to do with proving a function to be bijective.

The answer is (d). One should assume that $g \circ f(a_1) = g \circ f(a_2)$ and then prove that $a_1 = a_2$.

17. Define a relation R on $\mathbb{Z}$ by aRb if $a > b^2$. Then R has the following properties: (determine the most complete correct answer)

   (a) R is reflexive.
   (b) R is symmetric.
   (c) R is transitive.
   (d) R is reflexive and symmetric.
   (e) R is reflexive and transitive.
   (f) R is symmetric and transitive.
   (g) R is an equivalence relation.

The answer is (c). Note that $2R2$ is false, since $2 < 2^2$, so R is not reflexive. Also, $3R1$ but $1\not R3$, so R is not symmetric. Finally, if aRb and bRc, then $a > b^2$ and $b > c^2$. Since b is an integer, $b^2 \geq b$. Hence, $a > b^2 \geq b > c^2$, so $a > c^2$, and aRc. Thus R is transitive.

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18. Which of the following definitions is correct:

   (a) A prime number is any integer whose only integer divisors are itself and 1.
   (b) A function $f : A \to B$ is injective if every element of A maps to exactly one element of B.
   (c) A relation R on a nonempty set A is transitive if for $a, b, c \in A$, aRb and bRc imply cRa.
   (d) A set A is countably infinite if it is contained in the set $\mathbb{N}$ of natural numbers.
   (e) A sequence $\{a_n\}$ diverges to infinity if it does not converge to any finite limit.
   (f) None of the above–all of these definitions are incorrect.

The answer is (f). Note that: (a) prime numbers are positive, (b) this is the definition of a function, not an injective function, (c) the last thing should read aRc to be correct, (d) finite sets are not countably infinite, (e) the sequence $a_n = (-1)^n$ is a counterexample.

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19. In an ϵ-δ proof that

$$\lim_{x \to 1} 3x + 5 = 8,$$

which of the following is the largest δ that we can associate with a given $\epsilon > 0$.

(a) $\delta = 3$	(b) $\delta = 1/3$	(c) $\delta = \epsilon$	(d) $\delta = 3\epsilon$
(e) $\delta = \epsilon/3$	(f) $\delta = \epsilon/5$	(g) $\delta = \min(\epsilon, 1)$	(h) $\delta = 0$

The answer is (e). You find delta as follows.

$$\begin{aligned} |3x + 5 - 8| &< \varepsilon \\ 3|x - 1| &< \varepsilon \\ |x - 1| &< \varepsilon/3 \end{aligned}$$

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20. Which of the following facts is **not** always true about $\text{GCD}(a, b)$ for $a, b \in \mathbb{Z}$ with $b \neq 0$.

   (a) $\text{GCD}(a, b)$ divides both a and b.
   (b) $\text{GCD}(a, b) \geq 1$.
   (c) $\text{GCD}(a, b) \leq |b|$.
   (d) For any $q \in \mathbb{Z}$, $\text{GCD}(a, b) = \text{GCD}(a, qa + b)$.
   (e) $\text{GCD}(a, b)$ can be written as $ax + by$ with $x, y \in \mathbb{N}$.

The answer is (e). It is true that $\text{GCD}(a, b)$ can be written as $ax + by$ with $x, y \in \mathbb{Z}$, but, for instance, $\text{GCD}(5, 3)$ cannot be written as $5x + 3y$ with $x, y \in \mathbb{N}$.

Written Answer Section

21. Prove that if $a|b$ and $b|c$, then $a|c$, for integers a, b, c.

 Proof: Let $a, b, c \in \mathbb{Z}$. Assume that $a|b$ and $b|c$. Then $b = ak$ for some $k \in \mathbb{Z}$ and $c = bq$ for some $q \in \mathbb{Z}$. Thus $c = bq = akq$. Since $kq \in \mathbb{Z}$, $a|c$. $\square$

22. Find the largest integer that cannot be created from a (nonnegative) number of stamps of size 4 and 7. Then prove that all larger numbers can be so represented, by strong induction.

 The largest integer in 17. See the postage stamp problem on page 114 of the textbook for the format of the proof.

23. Prove/disprove: If $f : A \to B$ and $g : B \to C$ are functions and $g \circ f$ is injective, then f is injective. (Now change injective to surjective everywhere. Prove/disprove that f is surjective. Also prove/disprove that g is surjective.)

 See Theorem 26.12 of the textbook.

24. Prove for sets S and T that $S \subseteq T$ if and only if $S \cup T = T$.

 Proof: Assume that $S \subseteq T$. Let $x \in S \cup T$. Then $x \in T$ since $S \subseteq T$, so $S \cup T \subseteq T$. Let $x \in T$. Then $x \in S \cup T$. Thus $T \subseteq S \cup T$. That is, $S \cup T = T$. Now assume that $S \cup T = T$. Let $x \in S$. Then $x \in S \cup T$, so $x \in T$ since $S \cup T = T$. $\square$.

25. Prove that $\sum_{n=1}^{\infty} \frac{1}{2^n} = 1$.

 Proof: We work by induction to show that $\sum_{j=1}^{n} \frac{1}{2^j} = 1 - \left(\frac{1}{2^n}\right)$. Let $P(k)$ be the statement $\sum_{j=1}^{k} \frac{1}{2^j} = 1 - \left(\frac{1}{2^k}\right)$. (Base case) When $k = 1$, we have that $\sum_{j=1}^{k} \frac{1}{2^j} = 1/2 = 1 - 1/2 = 1 - \left(\frac{1}{2^k}\right)$. (Inductive step) Let $k \in \mathbb{N}$ and assume that $P(k)$ is true. Note that $\sum_{j=1}^{k+1} \frac{1}{2^j} = \frac{1}{2^{k+1}} + \sum_{j=1}^{k} \frac{1}{2^j} = \frac{1}{2^{k+1}} + 1 - \frac{1}{2^k} = 1 + \frac{1}{2^k}\left(\frac{1}{2} - 1\right) = 1 - \frac{1}{2^{k+1}}$. Thus by mathematical induction, $\sum_{j=1}^{n} \frac{1}{2^j} = 1 - \left(\frac{1}{2^n}\right)$ for all $n \in \mathbb{N}$. Hence, $\lim_{n\to\infty} \sum_{j=1}^{n} \frac{1}{2^j} = \lim_{n\to\infty} 1 - \frac{1}{2^n} = 1$. That is, $\sum_{j=1}^{\infty} \frac{1}{2^j} = 1$. $\square$

Other problems we could have put on the test include important theorems from the book like:

Prove that $(0, 1)$ is uncountable.

Prove that $|S| < |\mathcal{P}(S)|$ for any set S.

Prove there are infinitely many prime numbers.

and so forth...