# RED

Name: Key

Instructor: ****

# Math 290
# Sample Exam 2

Note that the first 10 questions are true–false. Mark A for true, B for false. Questions 11 through 20 are multiple choice. Mark the correct answer on your bubble sheet. Answers to the last five questions should be written directly on the exam, and should be written neatly and correctly. Questions 1 to 20 are worth 2.5 points each, and questions 21 to 25 are worth 10 points each.

## True–false questions

T 1. Let $\sim$ be the relation on $\mathbb{Z}$ defined by the rule $a \sim b$ if and only if $a - b$ is even. The relation $\sim$ is an equivalence relation with two equivalence classes.

F 2. There are two elements of $\mathbb{Z}_{11}$, both not $\overline{0}$, but their product is $\overline{0}$.

F 3. Given $S = \{1, 2, 3\}$, the set $R = \{(1, 1), (2, 2), (2, 3), (3, 2)\}$ is an equivalence relation on $S$.

T 4. Let $a, b \in \mathbb{Z}$. The smallest positive integer linear combination of $a$ and $b$ is $\mathrm{GCD}(a, b)$.

T 5. The greatest common divisor of 1073 and 1537 is 29.

F 6. Fix $n = (p_1 p_2 \cdots p_k) + 1$ where $p_1, p_2, \ldots, p_k$ are prime numbers. If $p$ is a prime number that divides $n$, then $p = p_i$ for some $i$.

T 7. We have $\overline{130 \cdot 2027 + 42} = \overline{2}$ in $\mathbb{Z}_{25}$.

F 8. For each natural number $n \geq 4$, we have $n! > 3^n$.

F 9. The coefficient of $x^5 y^3$ in $(2x - 3y)^8$ is $-\binom{8}{5} \cdot 3^5 \cdot 2^3$.

F 10. Every number $n \geq 2$ has at least two different prime factors.

## Multiple choice section

11. Which of the following claims is the most likely to require a proof by induction?
    (a) $x^2 - x = 5$ for some $x \in \mathbb{R}$.
    (b) The statement $P \Rightarrow Q$ is logically equivalent to $(\neg P) \vee Q$.
    (c) Then $n$th Fibonacci number $F_n$ is less than or equal to $2^{n-1}$, for each $n \geq 0$.
    (d) The number $\sqrt{6}$ is irrational.

12. Evaluate the proposed statement and proof.

    **Proposition**: For every $n \in \mathbb{N}$, it holds that $n^2 + 3 \geq 4n$.

    *Proof.* For the base case we note that when $n = 1$ we have $n^2 + 3 = 4 = 4n$. For the inductive step, suppose that $k^2 + 3 \geq 4k$ for some integer $k > 1$. Then we find

    $$(k + 1)^2 + 3 = k^2 + 2k + 1 + 3 \geq 2k + 2k + 4 = 4(k + 1).$$ ← does not hold when $k=1$.

    Thus, by the principle of mathematical induction, $n^2 + 3 \geq 4n$ for all $n \in \mathbb{N}$. □

    (a) The theorem is true, and the proof is correct.
    (b) The theorem is true, but the proof makes an error.
    (c) The theorem is false, but the proof is correct.
    (d) The theorem is false, and the proof makes an error.
    (e) None of the above.

13. Given an integer $a$, suppose that $a = 15q + 6$ for some $q \in \mathbb{Z}$. The greatest common divisor of $a$ and 15 is:

(a) 2.
(b) 3.
(c) 5.
(d) 6.
(e) 15.
(f) None of the above.

---

14. Let $S = \{1, 2, 3\}$. How many equivalence relations are there on $S$?

(a) 3.
(b) 4.
(c) 5.
(d) 6.
(e) 7.
(f) 8.

---

15. Which of the following is equal to the element $\overline{3}$ of $\mathbb{Z}_5$?
(a) $\{x^2 : x \in \mathbb{Z}\}$.
(b) $\{x + 3 : x \text{ is even}\}$.
(c) $\{3x + 16 : x \in \mathbb{Z}\}$.
(d) $\{3x : x = 5n \text{ for some } n \in \mathbb{Z}\}$.
(e) None of the other sets is equal to $\overline{3}$.

---

16. Evaluate the proposed result and proof.
**Proposition**: Let $n \in \mathbb{N}$. If $\overline{a}, \overline{b} \in \mathbb{Z}_n$ and $\overline{a} \cdot \overline{b} = \overline{0}$, then $\overline{a} = \overline{0}$ or $\overline{b} = \overline{0}$.

*Proof.* Suppose $a, b \in \mathbb{Z}$ with $\overline{a} \cdot \overline{b} = \overline{0}$ in $\mathbb{Z}_n$. Then $\overline{ab} = \overline{0}$, and hence $ab \equiv 0 \pmod{n}$. Thus $ab = kn$ for some $k \in \mathbb{Z}$. In other words $n|(ab)$. Hence $n|a$ or $n|b$, by Euclid's Lemma. Thus $\overline{a} = \overline{0}$, or $\overline{b} = \overline{0}$. □

(a) The proof correctly verifies that the proposition is true as stated.
(b) The proof would be correct, but assumes a premise that is not stated in the proposition, and the proposition is false.
(c) The proof is incorrect, because it is missing a case, but the proposition is true as stated.
(d) The proposition is true, but the proof (correctly) proves something else.

---

17. Define an equivalence relation on $\mathbb{Z}_4$ by $\overline{a} R \overline{b}$ if and only if $\overline{a}^2 = \overline{b}^2$. Which of the following is an equivalence class resulting from this relation?

(a) $\{\overline{0}, \overline{2}, \overline{3}\}$.
(b) $\{\overline{1}, \overline{2}\}$.
(c) $\{\overline{2}, \overline{3}\}$.
(d) $\{\overline{0}, \overline{2}\}$.
(e) $\emptyset$.
(f) $\mathbb{Z}_4$.
(g) None of the above.

---

18. Let $A = \{1, 2, 3, 4\}$, and let

$$R = \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,4), (3,1), (3,3), (3,4), (4,1), (4,2), (4,4)\}.$$

Which of the following best describes the properties of $R$ as a relation on $A$?
(a) $R$ is reflexive, symmetric, and transitive.
(b) $R$ is reflexive and symmetric but not transitive.
(c) $R$ is reflexive and transitive but not symmetric.
(d) $R$ is reflexive but neither symmetric nor transitive.
(e) $R$ is symmetric and transitive but not reflexive.
(f) $R$ is symmetric but neither reflexive nor transitive.
(g) $R$ is transitive but neither reflexive nor symmetric.
(h) $R$ is not reflexive, not symmetric, and not transitive.

19. How many relations are there on $S = \{1, 2, 3\}$?

(a) 6. (c) 25. (e) 120.
(b) 24. (d) 64. (f) 512.

---

20. Let $a, b, x, y \in \mathbb{Z}$. If $3 = ax + by$, and $3|a$ and $3|b$, then
(a) Both $a$ and $b$ must be relatively prime.
(b) Both $a$ and $y$ must be relatively prime.
(c) Both $x$ and $b$ must be relatively prime.
(d) Both $x$ and $y$ must be relatively prime.
(e) 3 is relatively prime to both $a$ and $b$.

---

**Written Answer Section**

21. Define five of the following seven terms (written in boldface) by completing the sentences. Only the first five defined terms will be graded.

(1) Given $n, k \in \mathbb{Z}$, we define the **binomial coefficient**, written $\binom{n}{k}$, as the number

(2) Two integers $a, b \in \mathbb{Z}$ are **relatively prime** if

(3) An **equivalence relation** on a set $A$ is a relation satisfying the three properties:

(4) If $\sim$ is an equivalence relation on a set $A$, then a **transversal** for $\sim$ is

(5) A relation $R$ is **reflexive** if

6) Given two nonzero integers $a, b \in \mathbb{Z}$, a **common divisor** is

7) Given an equivalence relation $\sim$ on a set $A$, and given $a \in A$, the **equivalence class** $[a]$ is the set

---

22. Let $x_1 = 1$, $x_2 = 5$, and $x_n = x_{n-1} + 2x_{n-2}$ for $n \geq 3$. Thus $x_3 = 7$, $x_4 = 17$, and so forth. Prove that, for every positive integer $n$,

$$x_n = 2^n + (-1)^n.$$

---

23. Find GCD$(-493, 391)$, and write the GCD as a linear combination of $-493$ and 391. (Make sure to box your two answers.)

---

24. Do each of the following:

(1) Write, in set-builder notation, one of the elements of $\mathbb{Z}_7$. Also give a transversal for $\mathbb{Z}_7$.

(2) Let $n \in \mathbb{N}$. Prove that congruence modulo $n$ is an equivalence relation on the set $\mathbb{Z}$.

---

25. For **BOTH** of the following statements, give a proof or disproof (clearly stating which one you are doing in each case).

(a) Let $a, b, c \in \mathbb{Z}$. If $a|bc$ and GCD$(a, b) = 1$, then $a|c$.

(b) Let $a, b, c \in \mathbb{Z}$. If $a|c$ and $b|c$, then $ab|c$.

21. (1) $\frac{n!}{k!(n-k)!}$

(2) $GCD(a,b) = 1$

(3) reflexive, symmetric, and transitive.

(4) a set which contains precisely one element from each equivalence class of $\sim$.

(5) for all $a \in A$, $aRa$.

(6) an integer $c \in \mathbb{Z}$ such that $c$ divides both $a$ and $b$.

(7) $[a] = \{ x \in A : a \sim x \}$

22. Proof by induction. <u>Base cases</u>:

$n=1$: $x_1 = 1 = 2^1 + (-1)^1$

$n=2$: $x_2 = 5 = 2^2 + (-1)^2$

Inductive step: Suppose now that for some $k \geq 2$ we have $x_p = 2^p + (-1)^p$ for all $1 \leq p \leq k$. Then:

$$x_{k+1} = x_k + 2x_{k-1} = \left(2^k + (-1)^k\right) + 2\left(2^{k-1} + (-1)^{k-1}\right)$$
$$= 2^k + (-1)^k + 2^k + 2(-1)^{k-1}$$
$$= 2^{k+1} + (-1)^k(1-2)$$
$$= 2^{k+1} + (-1)^{k+1}$$

Hence $x_n = 2^n + (-1)^n$ for all $n \geq 1$.

23. $493 = 391 + 102$

$391 = 3 \cdot 102 + 85$

$102 = 85 + 17$

$85 = 5 \cdot 17 \quad \Rightarrow \text{GCD}(-493, 391) = 17$

$$\Rightarrow \quad 17 = 102 - 85$$
$$= 102 - (391 - 3 \cdot 102)$$
$$= 4 \cdot 102 - 391$$
$$= 4(493 - 391) - 391$$
$$= 4 \cdot 493 - 5 \cdot 391$$

$\boxed{\text{GCD}(-493, 391) = 17}$ $\boxed{17 = 4 \cdot 493 - 5 \cdot 391}$

24. (1) $\bar{2} = \{2 + 7k : k \in \mathbb{Z}\} \in \mathbb{Z}_7$

Transversal: $\{0, 1, 2, 3, 4, 5, 6\}$

(2) (See Theorem 23.2 in the text book.)

25. (a) <u>Proof</u>: Suppose $a \mid bc$ and $GCD(a, b) = 1$.
Then

$$bc = ak \quad \text{for some } k \in \mathbb{Z}$$

and

$$xa + yb = 1 \quad \text{for some } x, y \in \mathbb{Z}.$$

Then

$$\begin{aligned} c \cdot (xa + yb) &= c \\ x \cdot ca + ybc &= c \\ xca + yak &= c \\ a(xc + yk) &= c \end{aligned}$$

and hence $a \mid c$. $\square$

(b) <u>Disproof</u>: Let $a = 2$, $b = 4$, $c = 4$. Then
$a \mid c$, $b \mid c$, but $ab \nmid c$.