Math 371, Midterm Exam #3 Study Guide

General information

(1) The exam will cover everything we have done since the last exam, including sections 7.1 to 7.9 (second edition) or 7.1-7.5 and 8.1-8.4 (third edition).
(2) Books, notes, and calculators will not be allowed.
(3) WARNING: this study guide is not meant to be exhaustive. Just because something is not on the study guide does not mean it will not be on the exam.

Basics

(1) Be able to do all homework problems.
(2) Know all the definitions discussed in the book, especially the definitions of
  - a group, an abelian group, a subgroup, a cyclic group
  - the center of a group, Cartesian products of groups, a simple group
  - group isomorphism and group homomorphism
  - left and right cosets
  - congruence modulo a subgroup
  - index of a subgroup
  - order of a group and an element
  - subgroup generated by a finite number of elements
  - quotient group
  - kernel of a group homomorphism

(3) Lots of examples of all the things we have discussed, especially:
  - Non-abelian groups: S_n, A_n, D_n, matrix groups.
  - Abelian groups: $\mathbb{Z}$, $\mathbb{Z}_n$, U_n.
  - An element of finite order contained in a group of infinite order.
  - Cyclic groups of all orders—both infinite and finite.
  - Groups which are not cyclic, including a (sub)group generated by two elements which is not cyclic.
  - A group with a non-trivial center.
  - A subgroup of an infinite group that has finite index.

Theorems you should know and be able to prove and use

- The First Isomorphism Theorem for groups (Theorem 7.42 in Ed.2, Theorem 8.20 in Ed.3).
- The center of a group is a subgroup (Theorem 7.12 in Ed.2, Theorem 7.13 in Ed.3).
- Lagrange's theorem: the order of a subgroup of a finite group divides the order of the group (Theorem 7.26 in Ed.2, Theorem 8.5 in Ed.3).
- The simple criterion for checking that a subset is a subgroup (Theorem 7.10 in Ed.2, Theorem 7.11 in Ed.3).
- Every ring is an abelian group under the addition of the ring.
- If R is a ring with identity, then the set of units of R is a group under multiplication of the ring.
- The identity element of a group is unique.
- Cancellation holds in a group; that is, $ab = ac$ or $ba = ca$ implies that $b = c$.
- In a group, inverses are unique.
- Every k-cycle in S_n has order k.
- Groups of prime order are cyclic.
- The kernel of a homomorphism $f : G \to H$ is a normal subgroup of G.
- A homomorphism is injective if and only if its kernel is trivial.
- If N is a normal subgroup of G, then the set G/N of all cosets of N in G forms a group with the product $(Na)(Nc) = N(ac)$ (the induced operation).
- If N is a normal subgroup of G, then there is a (canonical) surjective homomorphism $G \to G/N$.

Theorems you should be able to use

- Every permutation is either even or odd, but not both.
- Every subgroup of a cyclic group is cyclic.
- Disjoint cycles in S_n commute.
- Every permutation in S_n is the product of disjoint cycles.
- Every permutation is the product of transpositions.
- If N is a normal subgroup of G and K is a subgroup of G containing N, then K/N is a subgroup of G/N.
- Cayley's Theorem: every group is isomorphic to a group of permutations.
- The Third Isomorphism Theorem for groups.
- Equivalent conditions to being a normal subgroup (Theorem 7.34 in Ed.2, Theorem 8.11 in Ed.3).
- G/N is abelian if and only if $aba^{-1}b^{-1} \in N$ for all $a, b \in G$.

Sample problems

(1) Show that the group $\mathbb{Z}_5 \times \mathbb{Z}_2$ is cyclic, and that $\mathbb{Z}_6 \times \mathbb{Z}_2$ is not cyclic but is generated by two elements.
(2) Prove that inverses are unique in a group.
(3) Give an example of a non-abelian group G of order 24 and identify its center $Z(G)$.
(4) Are there any groups of order 3 which are not cyclic? If so, give an example; if not, prove it.
(5) Find a non-cyclic normal subgroup N of D_4 and determine what D_4/N is isomorphic to.
(6) List all subgroups of S_3 and show whether each is normal.
(7) Let K be a subgroup of a group G and let $b \in G$. Show that the set $b^{-1}Kb = \{b^{-1}kb : k \in K\}$ is a subgroup of G.
(8) Prove that A_n is a normal subgroup of S_n.
(9) Write the permutation $(147)(24)(3261)(45)$ as a product of disjoint cycles.
(10) Determine whether $(1423)(58)(679) \in S_{10}$ is even or odd.
(11) Prove that $H := \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \middle| \, a = \pm 1, b \in \mathbb{Z} \right\}$ is a subgroup of $GL(2, \mathbb{Q})$, the group of invertible 2 by 2 matrices with entries from $\mathbb{Q}$.
(12) Is there an element in S_4 of order 6? Prove that your answer is correct.
(13) Prove that there is no nontrivial group homomorphism from S_3 to $\mathbb{Z}_3$.
(14) Prove that $(\mathbb{Z} \times \mathbb{Z})/\langle(0, 1)\rangle$ is an infinite cyclic group.