TRACES OF SINGULAR MODULI, MODULAR FORMS, AND MAASS FORMS

By

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Abstract

We use Maass-Poincaré series to compute exact formulas for traces of singular moduli, and use these formulas to give a new proof of some identities of Zagier relating traces of singular moduli to the coefficients of certain half integral weight modular forms. These results imply a new proof of the infinite product isomorphism announced by Borcherds in his 1994 ICM lecture. The exact formulas derived for the traces also give an algorithm for computing the Hilbert class polynomial. We discuss p-adic properties of traces and the congruences that follow, and examine various generalizations of these traces, such as twisting by a genus character or varying the weight and level, proving various p-divisibility theorems. We conclude by proving a theorem about criteria for p-divisibility of class numbers of imaginary quadratic fields in terms of the p-divisibility of traces of singular moduli.

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Chapter 1

Introduction

1.1 Borcherds' isomorphism

We begin by looking at the infinite product expansion of the weight 12 cusp form

$$\Delta(z) = \frac{E_4(z)^3 - E_6(z)^2}{1728} = q - 24q^2 + 252q^3 - \cdots .$$

(Here, and throughout, $q = e^{2\pi i z}$.) Specifically, it is well known that

$$\Delta(z) = q \prod_{n=1}^{\infty} (1 - q^n)^{24}.$$

Looking at the weight 1/2 modular form

$$12\theta(z) = 12 + 24q + 24q^4 + 24q^9 + 24q^{16} + \cdots ,$$

we note that its constant term matches the weight of Δ and its coefficients match the infinite product exponents.

This correspondence is even more striking when another modular form, the Eisenstein series of weight 4, is considered, since its infinite product expansion

$$\begin{aligned} E_4(z) &= 1 + 240 \sum_{n=1}^{\infty} \sum_{v|n} v^3 q^n \\ &= (1-q)^{-240}(1-q^2)^{26760}(1-q^3)^{-4096240} \cdots = \prod_{n=1}^{\infty} (1-q^n)^{c(n)} \end{aligned}$$

has exponents that are not so easily described. Yet there exists a modular form $G(z)$ of weight $1/2$ given by

$$G(z) = \sum_{n=-3}^{\infty} b(n)q^n = q^{-3} + 4 - 240q + 26760q^4 + \cdots - 4096240q^9 + \cdots ,$$

where again the constant term matches the weight of $E_4(z)$ and the exponent $c(n)$ of $1-q^n$ in the product expansion of E_4 is equal to the coefficient $b(n^2)$ of q^{n^2} in the Fourier expansion of G.

These examples illustrate a famous theorem of Borcherds, where he proved that there is an isomorphism between the space of weakly holomorphic modular forms on $\Gamma_0(4)$ satisfying Kohnen's plus space condition and the space of integer weight meromorphic modular forms on $\mathrm{SL}_2(\mathbb{Z})$ with a Heegner divisor, integer coefficients, and leading coefficient 1. Recall that a such a modular form has a Heegner divisor if its zeros and poles occur only at the cusp at infinity and at CM points (imaginary quadratic irrational numbers in the upper half plane).

To describe this isomorphism, define the series $\tilde{H}(z)$ by

$$\tilde{H}(z) = -\frac{1}{12} + \sum_{1<d\equiv 0,3(4)} H(d)q^d = -\frac{1}{12} + \frac{q^3}{3} + \frac{q^4}{2} + q^7 + q^8 + \cdots ,$$

where $H(d)$ is the standard Hurwitz class number. For a weight $1/2$ weakly holomorphic modular form on $\Gamma_0(4)$ given by

$$f(z) = \sum_{\substack{d \geq d_0 \\ d\equiv 0,1(4)}} A(n)q^n,$$

define $\Psi(f(z))$ as

$$\Psi(f(z)) = q^{-h} \prod_{n=1}^{\infty} (1 - q^n)^{A(n^2)},$$

where h is the constant term of $f(z)\tilde{H}(z)$. Borcherds' theorem is as follows.

Theorem 1.1 ([3], Theorem 14.1). *The map* Ψ *is an isomorphism between the additive group of weakly holomorphic modular forms on* $\Gamma_0(4)$ *with integer coefficients* $A(n)$ *satisfying* $A(n) = 0$ *unless* $n \equiv 0, 1 \pmod 4$ *and the multiplicative group of integer weight modular forms on* $\mathrm{SL}_2(\mathbb{Z})$ *with Heegner divisor, integer coefficients, and leading coefficient 1. Under this isomorphism, the weight of* $\Psi(f(z))$ *is* $A(0)$*, and the multiplicity of the zero of* $\Psi(f(z))$ *at a Heegner point* z *of discriminant* $D < 0$ *is* $\sum_{n>0} A(Dn^2)$.

Borcherds proved this theorem as a consequence of his work on denominator formulas of infinite dimensional Lie algebras. In his paper, he asked whether a proof of this isomorphism existed that uses only modular forms on $SL_2(\mathbb{Z})$ and not automorphic forms on larger groups. Zagier answered this question in the affirmative in [27] by using traces of singular moduli. His proof has two parts: showing that these traces are coefficients of weakly holomorphic weight 1/2 modular forms on $\Gamma_0(4)$ ("modularity"), and establishing a striking duality between weakly holomorphic modular forms of weights 1/2 and 3/2 ("duality"). The results in this thesis are motivated by Zagier's proof of Borcherds' isomorphism.

1.2 Notation

To describe Zagier's work, we must define some notation. Let $j(z)$ be the usual modular function for $\mathrm{SL}_2(\mathbb{Z})$ defined by

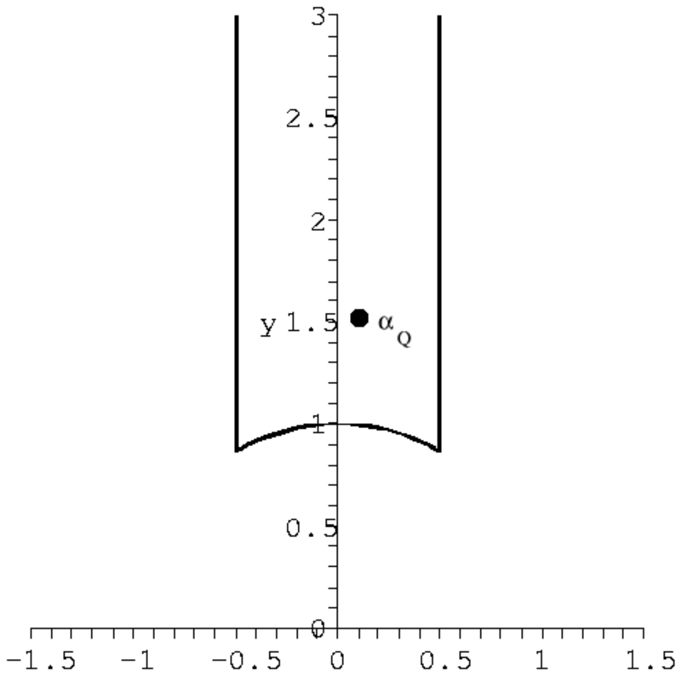
$$j(z) = \frac{E_4(z)^3}{\Delta(z)} = q^{-1} + 744 + 196884q + \cdots .$$

Let $d \equiv 0, 3 \pmod 4$ be a positive integer, so that $-d$ is a negative discriminant. Denote by $\mathcal{Q}_d$ the set of positive definite integral binary quadratic forms $Q(x, y) =$

$ax^2 + bxy + cy^2 = [a, b, c]$ with discriminant $-d = b^2 - 4ac$, including imprimitive forms (if such exist). We let α_Q be the unique complex number in the upper half plane $\mathfrak{H}$ which is a root of $Q(x, 1) = 0$.

Values of j at the points α_Q are known as *singular moduli.* For example, $j(i) = 1728$, $j\left(\frac{-1+\sqrt{-3}}{2}\right) = 0$, and $j(\frac{1+i\sqrt{15}}{2}) = \frac{-191025-85995\sqrt{5}}{2}$. Singular moduli are algebraic integers which play prominent roles in number theory. For example, Hilbert class fields of imaginary quadratic fields are generated by singular moduli, and isomorphism classes of elliptic curves with complex multiplication are distinguished by singular moduli. Because of the modularity of j, the singular modulus $j(\alpha_Q)$ depends only on the equivalence class of Q under the action of $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$, so we may choose Q such that α_Q is in the standard fundamental domain

$$\mathcal{F} = \{z = x + iy \in \mathfrak{H} : |z| \geq 1, -\frac{1}{2} \leq x < \frac{1}{2}, x \leq 0 \text{ if } |z| = 1\}.$$



We define the isotropy numbers $\omega_Q \in \{1, 2, 3\}$ as

$$\omega_Q = \begin{cases} 2 & \text{if } Q \sim_\Gamma [a, 0, a], \\ 3 & \text{if } Q \sim_\Gamma [a, a, a], \\ 1 & \text{otherwise;} \end{cases} \tag{1.1}$$

here ω_Q is the order of the stabilizer of Q in $\mathrm{PSL}_2(\mathbb{Z})$.

The Hurwitz-Kronecker class number $H(d)$ is the number of equivalence classes of forms of discriminant $-d$ under the action of Γ, weighted by ω_Q. Specifically, it is defined as

$$H(d) = \sum_{Q \in \mathcal{Q}_d/\Gamma} \frac{1}{\omega_Q}. \tag{1.2}$$

For fundamental discriminants $-d < -4$, this coincides with the standard class number $h(-d)$ of the imaginary quadratic field $\mathbb{Q}(\sqrt{-d})$.

Following Zagier, we define the trace of the singular moduli of discriminant $-d$ as

$$\mathrm{Tr}(d) = \sum_{Q \in \mathcal{Q}_d/\Gamma} \frac{j(\alpha_Q) - 744}{\omega_Q}. \tag{1.3}$$

Since we sum over all forms of discriminant $-d$, this is an actual algebraic trace.

The minimal polynomial of the singular moduli of discriminant $-d$ is known as the Hilbert class polynomial. This is a monic polynomial with integer coefficients of order $H(d)$; it generates the Hilbert class field (the maximal abelian unramified extension) of $\mathbb{Q}(\sqrt{-d})$. Calculating these polynomials is a classical difficult problem. We modify this polynomial slightly by adding a power of $1/\omega_Q$, defining

$$\mathcal{H}_d(X) = \prod_{Q \in \mathcal{Q}_d/\Gamma} (X - j(\alpha_Q))^{1/\omega_Q}.$$

By doing so, we obtain $H(d)$ and $\mathrm{Tr}(d)$ as the first two Fourier coefficients of the logarithmic derivative of $\mathcal{H}_d(j(z))$.

Zagier generalized these traces to describe the full Fourier expansion of this logarithmic derivative in the following manner. For a nonnegative integer m, let $j_m(z)$ be the unique holomorphic function on $\mathfrak{H}/\Gamma$ with a Fourier expansion beginning $q^{-m} + \mathcal{O}(q)$. For $m = 0$ this is the constant function 1, and for $m = 1$ it is $j(z) - 744$. Uniqueness follows easily from noting that the difference of any two such functions is a holomorphic cusp form of weight zero and is thus zero, and existence follows from noting that any such function can clearly be obtained as a monic polynomial in $j(z)$. The next few $j_m(z)$ are as follows.

$$j_2(z) = j(z)^2 - 1488j(z) + 159768 = q^{-2} + 42987520q + 40491909396q^2 + \dots$$

$$j_3(z) = j(z)^3 - 2232j(z)^2 + 1069956j(z) - 36866976 = q^{-3} + 2592899910q + \dots$$

$$\begin{aligned} j_4(z) &= j(z)^4 - 2976j(z)^3 + 2533680j(z)^2 - 561444608j(z) + 8507424792 \\ &= q^{-4} + 80983425024q + \dots \end{aligned}$$

We then define a generalized trace $\mathrm{Tr}_m(d)$ as

$$\mathrm{Tr}_m(d) = \sum_{Q \in \mathcal{Q}_d/\Gamma} \frac{j_m(\alpha_Q)}{\omega_Q}.$$

Note that for a fundamental discriminant $-d$, if we know all $\mathrm{Tr}_m(d)$ for all d up to the class number $h(-d)$, we can obtain the power sums

$$P_n(d) = \sum_{Q \in \mathcal{Q}_d/\Gamma} j(\alpha_Q)^n$$

as combinations of these traces and the coefficients of the $j_n(z)$. We can then use the Newton-Girard formulas

$$(-1)^m m S_m(d) + \sum_{k=1}^{m} (-1)^{k+m} P_k(d) S_{m-k}(d) = 0$$

to inductively obtain the symmetric functions $S_m(d)$ in the set $\{j(\alpha_Q) \colon Q \in \mathcal{Q}_d/\Gamma\}$, thus obtaining every coefficient of the Hilbert class polynomial $\mathcal{H}_d(X)$.

1.3 Zagier's results

Zagier showed that these traces $\mathrm{Tr}_m(d)$ appear as coefficients of certain weight $1/2$ weakly holomorphic modular forms on the group

$$\Gamma_0(4) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})\colon c \equiv 0 \pmod 4 \right\}.$$

A modular form is said to be weakly holomorphic if its poles are supported at cusps. To describe Zagier's forms more precisely, we let $M^!_{\lambda+\frac{1}{2}}$ be the space of weight $\lambda + \frac{1}{2}$ weakly holomorphic forms on $\Gamma_0(4)$ with Fourier coefficients satisfying the plus space condition congruences

$$f(z) = \sum_{(-1)^\lambda n \equiv 0,1 \pmod 4} a(n) q^n.$$

We will need to use Hecke operators on these spaces. For a modular form $f(z) = \sum a(n)q^n$ of weight $\lambda + \frac{1}{2}$ and an odd prime p, the half integral weight Hecke operator $T'(p^2)$ maps $f(z)$ to the modular form of weight $\lambda + \frac{1}{2}$ given by

$$f(z)|_{\lambda+\frac{1}{2}} T'(p^2) = \sum \left(a(p^2 n) + \left(\frac{(-1)^\lambda}{n} \right) \left(\frac{n}{p} \right) p^{\lambda-1} a(n) + p^{2\lambda-1} a(n/p^2) \right) q^n.$$

On $M^!_{\lambda+\frac{1}{2}}$, this formula also holds for $p = 2$ if we take $\left(\frac{n}{2}\right)$ to be 0 if n is even and $(-1)^{(n^2-1)/8}$ if n is odd. In the case $\lambda \leq 0$, this formula introduces nontrivial denominators, so we normalize by multiplying by $p^{1-2\lambda}$ so that our forms will still have integer coefficients, giving

$$T(p^2) = \begin{cases} p^{1-2\lambda} T'(p^2) & \text{if } \lambda \leq 0, \\ T'(p^2) & \text{otherwise.} \end{cases}$$

For each $0 \leq d \equiv 0, 3 \pmod 4$, we let $f_d(z)$ be the unique modular form in $M^!_{\frac{1}{2}}$ with

Fourier expansion

$$f_d(z) = q^{-d} + \sum_{0<D\equiv 0,1 \pmod 4} A(D,d)q^D.$$

Similarly, for each $0 < D \equiv 0,1 \pmod 4$, we define $g_D(z)$ to be the unique modular form in $M^!_{\frac{3}{2}}$ with Fourier expansion

$$g_D(z) = q^{-D} + B(D,0) + \sum_{0<d\equiv 0,3 \pmod 4} B(D,d)q^d.$$

For completeness, we define $A(M,N)$ and $B(M,N)$ to be zero if M or N is not an integer.

Uniqueness is clear, since there are no cuspforms of weight 1/2 or holomorphic modular forms of weight 3/2 on $\Gamma_0(4)$ satisfying the plus space condition. To see existence, we need only construct f_0, f_3, g_1, and g_4 and then note that for $d > 3$ or $D > 4$ we may multiply f_{d-4} or g_{D-4} by $j(4z)$ to get a form with the correct leading coefficient, and then add appropriate multiples of forms f'_d or g'_D for $0 \leq d' < d$ and $0 \leq D' < D$ to successively kill off the coefficients of other negative powers of q.

Obviously, f_0 is just the theta function $\theta(z)$, and a nontrivial linear combination of f_0 and f_3 is given by $(\theta(z)E'_{10}(4z) - 5\theta'(z)E_{10}(4z))/\Delta(4z)$. We also get

$$g_1(z) = \frac{\eta(z)^2}{\eta(2z)} \cdot \frac{E_4(4z)}{\eta(4z)^6} = q^{-1} - 2 + 248q^3 - 492q^4 + \cdots,$$

where $\eta(z) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n)$ is Dedekind's eta function, and g_4 is constructed by applying the weight 3/2 Hecke operator $T(4)$ to g_1. It is clear that the f_d form a basis for $M^!_{\frac{1}{2}}$ and the g_D form a basis for $M^!_{\frac{3}{2}}$. It is also clear from the construction that all of the $A(D,d)$ and $B(D,d)$ are integers.

The Fourier expansions of the first few f_d are as follows.

$$\begin{array}{lcllllll}
f_0(z) & = & 1 & +2q & +2q^4 & +0q^5 & +0q^8 & +\cdots, \\
f_3(z) & = & q^{-3} & -248q & +26752q^4 & -85995q^5 & +1707264q^8 & +\cdots, \\
f_4(z) & = & q^{-4} & +492q & +143376q^4 & +565760q^5 & +18473000q^8 & +\cdots, \\
f_7(z) & = & q^{-7} & -4119q & +8288256q^4 & -52756480q^5 & +5734772736q^8 & +\cdots, \\
f_8(z) & = & q^{-8} & +7256q & +26124256q^4 & +190356480q^5 & +29071392966q^8 & +\cdots.
\end{array}$$

Similarly, the first few g_D have the following Fourier expansions.

$$\begin{array}{llllll}
g_1(z) = q^{-1} - 2 & +248q^3 & -492q^4 & +4119q^7 & -7256q^8 & +\cdots, \\
g_4(z) = q^{-4} - 2 & -26752q^3 & -143376q^4 & -8288256q^7 & -26124256q^8 & +\cdots, \\
g_5(z) = q^{-5} + 0 & +85995q^3 & -565760q^4 & +52756480q^7 & -190356480q^8 & +\cdots, \\
g_8(z) = q^{-8} + 0 & -1707264q^3 & -18473000q^4 & -5734772736q^7 & -29071392966q^8 & +\cdots.
\end{array}$$

Comparing these expansions, it appears that, up to sign, the coefficients of the form f_d match the coefficients of q^d in the forms g_D. Zagier proved that this pattern of duality continues; namely, he showed that for all D and d,

$$A(D, d) = -B(D, d).$$

In fact, even more is true. For any integer $m \geq 1$, we may apply the weight 1/2 Hecke operator $|_{\frac{1}{2}}T(m^2)$ to f_d, and apply the weight 3/2 Hecke operator $|_{\frac{3}{2}}T(m^2)$ to g_D. We define $A_m(D, d)$ to be the coefficient of q^D in $f_d|_{\frac{1}{2}}T(m^2)$, and $B_m(D, d)$ to be the coefficient of q^d in $g_D|_{\frac{3}{2}}T(m^2)$. Note that $A_m(D, d), B_m(D, d) \in \mathbb{Z}$ from the definition of the Hecke operators. For completeness, define $A_m(D, d)$ and $B_m(D, d)$ to be 0 when D or d do not satisfy the appropriate congruence conditions or are not integers.

Because the f_d and g_D form bases for the spaces $M^!_{\frac{1}{2}}$ and $M^!_{\frac{3}{2}}$, one may easily use the definitions of the Hecke operators to compute

$$A_p(D,d) = pA_1(p^2D,d) + \left(\frac{D}{p}\right)A_1(D,d) + A_1\left(\frac{D}{p^2},d\right), \tag{1.4}$$

$$B_p(D,d) = B_1(D,p^2d) + \left(\frac{-d}{p}\right)B_1(D,d) + pB_1\left(D,\frac{d}{p^2}\right). \tag{1.5}$$

In addition, if $D = 1$, we have

$$A_m(1,d) = \sum_{n|m} nA_1(n^2,d) \tag{1.6}$$

and its generalization for fundamental discriminants D

$$A_m(D,d) = \sum_{n|m} n\left(\frac{D}{m/n}\right)A_1(n^2D,d). \tag{1.7}$$

Zagier proved the following theorem, the "duality" part of his proof of Borcherds' theorem.

Theorem 1.2 (Duality; [27], Theorem 5(iii)). *For all m, D, and d, we have*

$$A_m(D,d) = -B_m(D,d).$$

The other major part of Zagier's proof of Borcherds' isomorphism was establishing the modularity of the generalized traces of singular moduli. Specifically, Zagier proved that the $\mathrm{Tr}_m(d)$ appear as coefficients of the form g_1 in the following manner.

Theorem 1.3 (Modularity; [27], Theorem 5(ii)). *For all m and d, we have*

$$\mathrm{Tr}_m(d) = -B_m(1,d).$$

To see how Zagier proved Borcherds' isomorphism from these theorems, rewrite Theorem 1.1 as

$$\mathcal{H}_d(j(z)) = q^{-H(d)} \prod_{n=1}^{\infty} (1-q^n)^{A_1(n^2,d)},$$

noting that $\mathcal{H}_d(j(z))$ is a weight zero modular function on $\mathrm{SL}_2(\mathbb{Z})$ with zeros at each Heegner point of discriminant $-d$ and a pole of order $H(d)$ at infinity. Given Theorems 1.2 and 1.3 and equation (1.6), this may be proved by proving that

$$\mathcal{H}_d(j(z)) = q^{-H(d)} \exp\left(-\sum_{m=1}^{\infty} \mathrm{Tr}_m(d)\frac{q^m}{m}\right).$$

This statement follows from the definitions of $\mathcal{H}_d(X)$ and $\mathrm{Tr}_m(d)$ and the identity

$$j(\alpha) - j(z) = q^{-1} \exp\left(-\sum_{m=1}^{\infty} j_m(\alpha)\frac{q^m}{m}\right) \quad (\mathrm{Im}(\alpha) \gg 0).$$

The identity is proved by taking the logarithmic derivative with respect to α of each side and noting that for α with sufficiently large imaginary part, each side is a Γ-invariant meromorphic function of z vanishing at infinity whose only pole in $\mathcal{F}$ is a simple pole of residue 1 at $z = \alpha$.

In this thesis, we give a more direct proof of Zagier's modularity theorem, thus giving a new proof of Borcherds' isomorphism. This new proof uses Maass-Poincaré series to build the functions g_D, giving exact formulas for the coefficients and thus for traces of singular moduli; such exact formulas give a direct algorithm for computing Hilbert class polynomials. In addition, we examine the p-adic properties of traces of singular moduli. We consider several generalizations of these traces, such as twisting by a genus character or passing to higher levels or different weights, and obtain several theorems in these cases. We conclude by proving a theorem about criteria for p-divisibility of class numbers of imaginary quadratic fields in terms of the p-divisibility of traces of singular moduli.

Chapter 2

Maass-Poincaré Series

In this chapter we develop Maass-Poincaré series and use them to construct the modular forms g_D, giving exact formulas for coefficients of these forms. The ideas in this chapter first appeared in [8], as joint work with Jan Bruinier and Ken Ono.

2.1 Results and applications

It is a natural question to investigate the asymptotics of $\mathrm{Tr}(d)$ as d grows. In fact, these asymptotics are closely related to the interesting classical observation that

$$e^{\pi\sqrt{163}} = 262537412640768743.99999999999925\ldots \tag{2.1}$$

is very close to an integer. To see this, recall that a positive definite binary quadratic form $Q = [A, B, C]$ is classically defined to be *reduced* if α_Q is in the standard fundamental domain $\mathcal{F}$, or equivalently if $|B| \leq A \leq C$, and $B \geq 0$ if either $|B| = A$ or $A = C$. If $-d < -4$ is a fundamental discriminant, there are $H(d)$ such reduced forms of discriminant $-d$. The set $\mathcal{Q}_d^{\mathrm{red}}$ of reduced forms of discriminant $-d$ is a complete set of representatives for $\mathcal{Q}_d/\Gamma$. It follows easily from $\mathrm{Im}(\alpha_Q) \geq \frac{\sqrt{3}}{2}$ that $1 \leq A \leq \sqrt{d/3}$ for reduced forms.

If we define

$$G^{\mathrm{red}}(d) = \sum_{Q \in \mathcal{Q}_d^{\mathrm{red}}} e^{\pi B i/A} e^{\pi\sqrt{d}/A}$$

to be the contribution to $\mathrm{Tr}(d)$ of the q^{-1} term in the expansion of $j(z)$, it follows that $\mathrm{Tr}(d) - G^{\mathrm{red}}(d)$ should be "small". This is illustrated by (2.1), where $H(163) = 1$ and $\mathrm{Tr}(163) = 262537412640768744$.

Remark. It turns out that $d = 163$ is the largest discriminant such that $\mathbb{Q}(\sqrt{d})$ has class number one; the relevant quadratic form is $[1, 1, 41]$. Since $A = 1$, we get a very good estimate, as the next term in the expansion of $j(\alpha_Q)$ is $196884 \cdot e^{-\pi\sqrt{163}/1} \approx 7.499 \cdot 10^{-13}$. For forms with $A > 1$, though, the term $196884 \cdot e^{-\pi\sqrt{d}/A}$ will not be as close to zero. Thus, we will not necessarily get more instances of "almost integers" by taking large discriminants of some specified class number, since many of the quadratic forms will have $A > 1$.

We next examine the average value

$$\frac{\mathrm{Tr}(d) - G^{\mathrm{red}}(d)}{H(d)}.$$

If $d = 1931, 2028$ and 2111, then we have

$$\frac{\mathrm{Tr}(d) - G^{\mathrm{red}}(d)}{H(d)} = \begin{cases} 11.981\ldots & \text{if } d = 1931, \\ -24.483\ldots & \text{if } d = 2028, \\ -13.935\ldots & \text{if } d = 2111. \end{cases}$$

These values are small, but do not seem to be uniform. However, if we slightly perturb these numbers by including certain non-reduced quadratic forms, a clearer picture emerges. For positive integers A, define

$$\mathcal{Q}^{\mathrm{old}}_{A,d} = \{Q = (A, B, C) : \text{ non-reduced with discriminant } -d \text{ and } |B| \leq A\}. \quad (2.2)$$

Additionally, define $G^{\text{old}}(d)$ by

$$G^{\text{old}}(d) = \sum_{\substack{\sqrt{d}/2 \le A \le \sqrt{d/3} \\ Q \in \mathcal{Q}^{\text{old}}_{A,d}}} e^{\pi Bi/A} \cdot e^{\pi\sqrt{d}/A}. \tag{2.3}$$

The forms counted in $G^{\text{old}}(d)$ are forms Q of discriminant $-d$ for which α_Q is in the region bounded by the lower boundary of $\mathcal{F}$ and the horizontal line connecting the two endpoints of that boundary.

For the values of d we examined before, we have

$$\frac{\text{Tr}(d) - G^{\text{red}}(d) - G^{\text{old}}(d)}{H(d)} = \begin{cases} -24.672\ldots & \text{if } d = 1931, \\ -24.483\ldots & \text{if } d = 2028, \\ -23.458\ldots & \text{if } d = 2111. \end{cases}$$

As $-d \to -\infty$, it appears that these values converge to -24. Duke [10] has recently proved a reformulation of this, implying the following theorem.

Theorem 2.1 ([10], Theorem 1). *As $-d$ ranges over negative fundamental discriminants, we have*

$$\lim_{-d \to -\infty} \frac{\text{Tr}(d) - G^{\text{red}}(d) - G^{\text{old}}(d)}{H(d)} = -24.$$

Duke's theorem depends on methods developed in [11] concerning the distribution of CM points. The constant -24 is obtained by characterizing limits of this type in terms of values of Atkin's inner product, which can then be directly evaluated.

By constructing the modular form g_1 using Maass-Poincaré series, we obtain an exact formula for $\text{Tr}(d)$ in terms of Kloosterman sums and $24H(d)$, giving a natural explanation for Theorem 2.1. This formula is the $m = 1$ case of a general family of formulas for the traces $\text{Tr}_m(d)$.

To state these formulas, we need some notation. For odd integers v, define ε_v by

$$\varepsilon_v = \begin{cases} 1 & \text{if } v \equiv 1 \pmod 4, \\ i & \text{if } v \equiv 3 \pmod 4. \end{cases} \tag{2.4}$$

We let $e(x) = e^{2\pi i x}$ throughout. For $k \in \frac{1}{2}\mathbb{Z}$, define the generalized Kloosterman sum

$$K_k(m,n,c) = \sum_{v\ (c)^*} \left(\frac{c}{v}\right)^{2k} \varepsilon_v^{2k} e\left(\frac{m\bar{v} + nv}{c}\right). \tag{2.5}$$

Here v runs through the primitive residue classes modulo c, and $\bar{v}$ is the multiplicative inverse of v modulo c. In addition, we define the function δ_{odd} on the integers by

$$\delta_{\text{odd}}(v) = \begin{cases} 1 & \text{if } v \text{ is odd}, \\ 0 & \text{otherwise}. \end{cases}$$

We then have the following formulas for the $\mathrm{Tr}_m(d)$.

Theorem 2.2. *If m is a positive integer and $-d < 0$ is a discriminant, then*

$$\mathrm{Tr}_m(d) = -\sum_{n|m} nB_1(n^2,d),$$

where $B_1(n^2,d)$ is the integer given by

$$B_1(n^2,d) = 24H(d) - (1+i) \sum_{\substack{c>0 \\ c\equiv 0\ (4)}} (1+\delta_{\text{odd}}(c/4)) \frac{K_{3/2}(-n^2,d,c)}{n\sqrt{c}} \sinh\left(\frac{4\pi n}{c}\sqrt{d}\right).$$

These formulas for the $\mathrm{Tr}_m(d)$ thus give an algorithm for computing Hilbert class polynomials for fundamental discriminants $-d$, as was discussed in the previous chapter. Some care must be taken, though, as the series for the integers $B_1(n^2,d)$ do not converge rapidly.

Proof of Theorem 2.2. Combining Zagier's modularity and duality theorems (Theorems 1.3 and 1.2) with equation (1.6), we easily see that

$$\mathrm{Tr}_m(d) = -\sum_{n|m} nB_1(n^2, d).$$

Theorem 2.2 then follows from the following theorem, which we will prove later in this chapter by constructing the functions g_D using Maass-Poincaré series.

Theorem 2.3. *Let D be a positive integer with $D \equiv 0, 1 \pmod 4$. Then the Fourier coefficient $B_1(D, n)$ with positive index n of the function g_D, where $n \equiv 0, 3 \pmod 4$, is given by*

$$B(D,n) = 24\delta_{\Box,D}H(n) - (1+i)\sum_{\substack{c>0\\ c\equiv 0\ (4)}}(1+\delta_{\mathrm{odd}}(c/4))\frac{K_{3/2}(-D,n,c)}{\sqrt{cD}}\sinh\left(\frac{4\pi}{c}\sqrt{Dn}\right).$$

Here $\delta_{\Box,D} = 1$ if D is a square, and $\delta_{\Box,D} = 0$ otherwise.

Substituting n^2 for D in this theorem, Theorem 2.2 follows immediately. □

2.2 Weakly holomorphic Poincaré series

Here we recall and derive results on non-holomorphic Poincaré series of half-integral weight. We use these half-integral weight series to construct certain weak Maass forms, and we use these forms to describe Zagier's weakly holomorphic modular forms

$$g_D(z) = q^{-D} + B_1(D,0) + \sum_{0<d\equiv 0,3 \pmod 4} B_1(D,d)q^d \in M^!_{\frac{3}{2}},$$

thereby obtaining exact formulas for the coefficients $B_1(D,d)$.

We begin by recalling facts about modular forms of half-integral weight. Here $\mathfrak{H}$ is the complex upper half plane, and $z = x + iy$ is the standard variable on $\mathfrak{H}$. Let $\mathfrak{G}$ be the group of pairs $(A, \phi(z))$, where $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2^+(\mathbb{R})$ and ϕ is a holomorphic function on $\mathfrak{H}$ satisfying

$$|\phi(z)| = (\det A)^{-1/4}|cz + d|^{1/2}.$$

The group law is given by

$$(A_1, \phi_1(z))(A_2, \phi_2(z)) = (A_1A_2, \phi_1(A_2z)\phi_2(z)).$$

Throughout, suppose that $k \in \frac{1}{2}\mathbb{Z}$. The group $\mathfrak{G}$ acts on functions $f \colon \mathfrak{H} \to \mathbb{C}$ by

$$(f \mid_k (A, \phi))(z) = \phi(z)^{-2k} f(Az).$$

For $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(4)$, let

$$j(\gamma, z) = \left(\frac{c}{d}\right)\varepsilon_d^{-1}\sqrt{cz + d}$$

be the automorphy factor for the classical Jacobi theta function $\theta(z) = \sum_{n=-\infty}^{\infty} q^{n^2}$. Here $\sqrt{z}$ is the principal branch of the holomorphic square root, and ε_d is given by (2.4).

The map

$$\gamma \mapsto \tilde{\gamma} = (\gamma, j(\gamma, z))$$

defines a group homomorphism $\Gamma_0(4) \to \mathfrak{G}$. For convenience, if $\gamma \in \Gamma_0(4)$, we write $f \mid_k \gamma$ instead of $f \mid_k \tilde{\gamma}$.

Let $m \in \mathbb{Z}$ and $\varphi^0 \colon \mathbb{R}_{>0} \to \mathbb{C}$ be a function satisfying

$$\varphi^0(y) = O(y^\alpha), \qquad y \to 0, \tag{2.6}$$

for some $\alpha \in \mathbb{R}$. Then $\varphi(z) = \varphi^0(y)e(mx)$ is a function on $\mathfrak{H}$ which is invariant under the action of the subgroup $\Gamma_\infty = \{\left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right) : n \in \mathbb{Z}\}$ of $\Gamma_0(4)$. We consider the Poincaré series at the cusp ∞ of weight k for the group $\Gamma_0(4)$ given by

$$F(z, \varphi) = \frac{1}{2} \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(4)} (\varphi \mid_k \gamma)(z). \tag{2.7}$$

By comparing with an Eisenstein series, one shows that this series converges locally uniformly absolutely for $k > 2 - 2\alpha$. Hence, it is a $\Gamma_0(4)$-invariant function on $\mathfrak{H}$.

Let $c \in P^1(\mathbb{Q})$ be a cusp, and suppose that $(A, \phi) \in \mathfrak{G}$ is chosen so that $A \in \mathrm{SL}_2(\mathbb{Z})$ and $A\infty = c$. We say that a function $f(z)$ has *moderate growth* at c if there is a $C \in \mathbb{R}$ for which, as $y \to \infty$, we have

$$(f \mid_k (A, \phi))(z) = O(y^C).$$

Proposition 2.4. *Let φ be as above and assume that $k > 2 - 2\alpha$. Near the cusp at ∞, the function $F(z, \varphi) - \varphi(z)$ has moderate growth. Near the other cusps, $F(z, \varphi)$ has moderate growth. If $F(z, \varphi)$ is twice continuously differentiable, then it has the locally uniformly absolutely convergent Fourier expansion*

$$F(z, \varphi) = \varphi(z) + \sum_{n \in \mathbb{Z}} a(n, y) e(nx),$$

where

$$a(n, y) = \sum_{\substack{c>0 \\ c \equiv 0 \ (4)}}^{\infty} c^{-k} K_k(m, n, c) \int_{-\infty}^{\infty} z^{-k} \varphi^0 \left(\frac{y}{c^2 |z|^2} \right) e\left(-\frac{mx}{c^2 |z|^2} - nx \right) dx. \tag{2.8}$$

Proof of Proposition 2.4. The assertion is obtained by standard arguments. For completeness, we sketch how the Fourier expansion is calculated. By definition, we have

$$a(n, y) = \int_0^1 (F(z, \varphi) - \varphi(z))\, e(-nx)\, dx.$$

Inserting the definition of $F(z, \varphi)$ and applying Poisson summation, we obtain

$$a(n, y) = \sum_{\substack{\gamma \in \Gamma_\infty \backslash \Gamma_0(4) / \Gamma_\infty \\ c(\gamma) > 0}} \int_{-\infty}^{\infty} (\varphi \mid_k \gamma)(z) e(-nx)\, dx.$$

Using $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) z = \frac{a}{c} - \frac{1}{c^2(z+d/c)}$ and $\varphi(z) = \varphi^0(y)e(mx)$, we find that $a(n,y)$ is equal to

$$\sum_{\substack{\gamma\in\Gamma_\infty\backslash\Gamma_0(4)/\Gamma_\infty \\ c(\gamma)>0}} \left(\frac{c}{d}\right)^{2k} \varepsilon_d^{2k} c^{-k} \int_{-\infty}^{\infty} (z+d/c)^{-k} \varphi\left(\frac{a}{c} - \frac{1}{c^2(z+d/c)}\right) e(-nx)dx$$

$$= \sum_{c>0} \sum_{d\ (c)^*} \left(\frac{c}{d}\right)^{2k} \varepsilon_d^{2k} e\left(\frac{m\bar{d}+nd}{c}\right) c^{-k} \int_{-\infty}^{\infty} z^{-k} \varphi^0\left(\frac{y}{c^2|z|^2}\right) e\left(-\frac{mx}{c^2|z|^2} - nx\right) dx.$$

This yields the assertion. □

Particularly important Poincaré series are obtained by letting φ^0 be certain Whittaker functions. Let $M_{\nu,\mu}(z)$ and $W_{\nu,\mu}(z)$ be the usual Whittaker functions as defined on page 190 of Chapter 13 in [23]. Following [6] Chapter 1.3, for $s \in \mathbb{C}$ and $y \in \mathbb{R} - \{0\}$ we define

$$\mathcal{M}_s(y) = |y|^{-k/2} M_{k/2\,\mathrm{sgn}(y),\, s-1/2}(|y|), \tag{2.9}$$

$$\mathcal{W}_s(y) = |y|^{-k/2} W_{k/2\,\mathrm{sgn}(y),\, s-1/2}(|y|). \tag{2.10}$$

The functions $\mathcal{M}_s(y)$ and $\mathcal{W}_s(y)$ are holomorphic in s. Later we will be interested in certain special s-values. For $y > 0$, we have

$$\mathcal{M}_{k/2}(-y) = y^{-k/2} M_{-k/2,\, k/2-1/2}(y) = e^{y/2}, \tag{2.11}$$

$$\mathcal{W}_{1-k/2}(y) = y^{-k/2} W_{k/2,\, 1/2-k/2}(y) = e^{-y/2}. \tag{2.12}$$

If m is a non-zero integer, then the function

$$\varphi_{m,s}(z) = \mathcal{M}_s(4\pi m y) e(mx)$$

is an eigenfunction of the weight k hyperbolic Laplacian

$$\Delta_k = -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + iky\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right), \tag{2.13}$$

and has eigenvalue $s(1-s)+(k^2-2k)/4$. It satisfies $\varphi_{m,s}(z) = O(y^{\mathrm{Re}(s)-k/2})$ as $y \to 0$. Consequently, the corresponding Poincaré series

$$F_m(z,s) = F(z, \varphi_{m,s}) \tag{2.14}$$

converges for $\mathrm{Re}(s) > 1$, and it defines a $\Gamma_0(4)$-invariant eigenfunction of the Laplacian. In particular, $F_m(z,s)$ is real analytic.

Proposition 2.5. *If m is a negative integer, then the Poincaré series $F_m(z,s)$ has the Fourier expansion*

$$F_m(z,s) = \mathcal{M}_s(4\pi m y)e(mx) + \sum_{n\in\mathbb{Z}} c(n,y,s)e(nx),$$

where the coefficients $c(n,y,s)$ are given by

$$\begin{cases} \dfrac{2\pi i^{-k}\Gamma(2s)}{\Gamma(s-k/2)}\left|\dfrac{n}{m}\right|^{\frac{k-1}{2}} \displaystyle\sum_{\substack{c>0\\ c\equiv 0\ (4)}} \dfrac{K_k(m,n,c)}{c} J_{2s-1}\left(\dfrac{4\pi}{c}\sqrt{|mn|}\right)\mathcal{W}_s(4\pi n y), & n<0, \\ \dfrac{2\pi i^{-k}\Gamma(2s)}{\Gamma(s+k/2)}\left|\dfrac{n}{m}\right|^{\frac{k-1}{2}} \displaystyle\sum_{\substack{c>0\\ c\equiv 0\ (4)}} \dfrac{K_k(m,n,c)}{c} I_{2s-1}\left(\dfrac{4\pi}{c}\sqrt{|mn|}\right)\mathcal{W}_s(4\pi n y), & n>0, \\ \dfrac{4^{1-k/2}\pi^{1+s-k/2}i^{-k}|m|^{s-k/2}y^{1-s-k/2}\Gamma(2s-1)}{\Gamma(s+k/2)\Gamma(s-k/2)} \displaystyle\sum_{\substack{c>0\\ c\equiv 0\ (4)}} \dfrac{K_k(m,0,c)}{c^{2s}}, & n=0. \end{cases}$$

Here $J_\nu(z)$ and $I_\nu(z)$ denote the usual Bessel functions as defined in Chapter 9 of [23]. The Fourier expansion defines an analytic continuation of $F_m(z,s)$ to $\mathrm{Re}(s) > 3/4$.

Proof. In view of Proposition 2.4, it suffices to compute the integral

$$\begin{aligned} &\int_{-\infty}^{\infty} z^{-k}\mathcal{M}_s\left(4\pi m\frac{y}{c^2|z|^2}\right) e\left(-\frac{mx}{c^2|z|^2} - nx\right)dx \\ &= (4\pi|m|y)^{-k/2}c^k i^{-k} \\ &\quad\times \int_{-\infty}^{\infty}\left(\frac{y-ix}{y+ix}\right)^{-k/2} M_{-k/2,\,s-1/2}\left(\frac{4\pi|m|y}{c^2(x^2+y^2)}\right) e\left(-\frac{mx}{c^2(x^2+y^2)} - nx\right)dx. \end{aligned}$$

The latter integral equals the integral I in [6] (1.40). It is evaluated on [6] p. 33, and inserting it above yields the asserted formula for $c(n, y, s)$. $\square$

By noticing that the Poincaré series $F_m(z, s)$ occur as the Fourier coefficients of the automorphic resolvent kernel (also referred to as automorphic Green function) for $\Gamma_0(4)$, and by using the spectral expansion of the automorphic resolvent kernel, one finds that $F_m(z, s)$ has a meromorphic continuation in s to the whole complex plane. For $\mathrm{Re}(s) > 1/2$, it has simple poles at points of the discrete spectrum of Δ_k.

For the special s-values $k/2$ and $1 - k/2$, the function $F_m(z, s)$ is annihilated by Δ_k. As we will see below, it is actually often holomorphic in z. Consequently, these special values are of particular interest.

If $k \leq 1/2$ then $F_m(z, s)$ is holomorphic in s near $s = 1 - k/2$, and one can consider $F_m(z, 1 - k/2)$. This was done in [6] Chapter 1.3 in a slightly different setting, and has since been done in [5]. Here we are mainly interested in the case that $k \geq 3/2$ (later in particular in $k = 3/2$). Then $F_m(z, s)$ is holomorphic in s near $s = k/2$.

A *weak Maass form* of weight k for the group $\Gamma_0(4)$ is a smooth function $f\colon \mathfrak{H} \to \mathbb{C}$ such that

$$f \mid_k \gamma = f,$$

for all $\gamma \in \Gamma_0(4)$, and $\Delta_k f = 0$, with at most linear exponential growth at all cusps (see [7] Section 3). If such an f is actually holomorphic on $\mathfrak{H}$, it is called a *weakly holomorphic modular form*. It is then meromorphic at the cusps.

Theorem 2.6. *Assume the notation above.*

1. *If $k \geq 2$, then $F_m(z, k/2)$ is a weakly holomorphic modular form of weight k for*

the group $\Gamma_0(4)$. *The Fourier expansion at the cusp* ∞ *is given by*

$$F_m(z, k/2) = e(mz) + \sum_{n>0} c(n, y, k/2)e(nx),$$

where for $n > 0$ *we have*

$$c(n, y, k/2) = 2\pi i^{-k} \left|\frac{n}{m}\right|^{\frac{k-1}{2}} \sum_{\substack{c>0 \\ c\equiv 0\ (4)}} \frac{K_k(m, n, c)}{c} I_{k-1}\left(\frac{4\pi}{c}\sqrt{|mn|}\right) e^{-2\pi n y}. \tag{2.15}$$

At the other cusps, $F_m(z, k/2)$ *is holomorphic (and actually vanishes).*

2. *If* $k = 3/2$, *then* $F_m(z, k/2)$ *is a weak Maass form of weight* k *for the group* $\Gamma_0(4)$. *The Fourier coefficients* $c(n, y, k/2)$ *with positive index* n *are still given by* (2.15). *Near the cusp* ∞ *the function* $F_m(z, k/2) - e(mz)$ *is bounded. Near the other cusps the function* $F_m(z, k/2)$ *is bounded.*

Proof. If $k \geq 2$, the assertion immediately follows from Propositions 2.4 and 2.5. For the computation of the Fourier expansion, we notice that the sums over c in the formula of Proposition 2.5 converge absolutely by the Weil bound for Kloosterman sums. (For $k > 2$, one can actually argue more directly by noticing that $F_m(z, k/2) = F(z, e(mz))$ converges absolutely.)

If $k = 3/2$, then Proposition 2.4 and the discussion preceding the theorem imply that $F_m(z, k/2)$ is a weak Maass form and has the claimed growth near the cusps. The formula for the Fourier coefficients with positive index follows by analytic continuation using the fact that (2.15) converges. Convergence of such series is well known if one replaces the $I_{1/2}$-Bessel function by the $J_{1/2}$-Bessel function (for example, see [11, 15]). Since $J_{1/2}(1/x) \sim I_{1/2}(1/x)$ as $x \to +\infty$, the convergence of (2.15) follows. $\square$

Remark. Notice that if $k = 3/2$, $m = -a^2$, and $n = -b^2$, where $a, b \in \mathbb{Z}\backslash\{0\}$, then the series over c occurring in the formula for $c(n, y, s)$ in Proposition 2.5 diverges at $s = k/2$. The corresponding singularity cancels with the zero of $\Gamma(s - k/2)^{-1}$.

There is an anti-linear differential operator ξ_k that takes weak Maass forms of weight k to weakly holomorphic modular forms of weight $2 - k$ (see Proposition 3.2 of [7]). If $f(z)$ is a weak Maass form of weight k, then by definition

$$\xi_k(f)(z) = 2iy^k \overline{\tfrac{\partial}{\partial \bar{z}} f(z)}. \tag{2.16}$$

In addition, this operator has the property that $\ker(\xi_k)$ is the subset of weight k weak Maass forms which are weakly holomorphic modular forms (see Proposition 3.2 of [7]). Consequently, if $k \geq 2$, then Theorem 2.6 implies that $\xi_k(F_m(z, k/2)) = 0$. However, the situation is quite different when $k = 3/2$.

Proposition 2.7. *If $k = 3/2$ and $c(0)$ is chosen so that the constant coefficient of the Poincaré series $F_m(z, k/2)$ is given by*

$$c(0, y, k/2) = c(0)y^{1-k},$$

then

$$\xi_k(F_m(z, k/2)) = \frac{1}{2}c(0)\theta(z).$$

Proof. The assertion of Theorem 2.6 on the growth at the cusps of $F_m(z, k/2)$ implies that $\xi_k(F_m(z, k/2))$ is actually a holomorphic modular form of weight $2 - k = 1/2$ for the group $\Gamma_0(4)$. Hence it has to be a multiple of $\theta(z)$. By comparing the constant terms one obtains the factor of proportionality. $\square$

2.3 Projection to Kohnen's plus space

It is our aim to relate the generating functions for Zagier's traces of singular moduli to the Poincaré series of weight $k = 3/2$. More generally, we shall describe all of Zagier's functions $g_D(z) \in M^!_{3/2}$ in terms of the Poincaré series $F_{-D}(z, k/2)$.

One easily checks that $g_D(z)$ has, in general, poles at all cusps of $\Gamma_0(4)$, while the singularities of $F_m(z, k/2)$ are supported at the cusp infinity. Consequently, we also have to consider Poincaré series at the other cusps of $\Gamma_0(4)$ and take suitable linear combinations. This can be done in a quite conceptual (and automatic) way by applying a projection operator to the Kohnen plus-space.

Throughout this subsection, assume that $1 \leq \lambda \in \mathbb{Z}$ and $k = \lambda + 1/2$. Kohnen (see p. 250 of [20]) constructed a projection operator $|_k$ pr from the space of modular forms of weight k for $\Gamma_0(4)$ to the subspace of those forms satisfying the plus-space condition. It is defined by

$$f \mid_k \mathrm{pr} = \frac{1}{3} f + (-1)^{\lfloor(\lambda+1)/2\rfloor} \frac{1}{3\sqrt{2}} \sum_{\nu\ (4)} f \mid_k B\tilde{A}_\nu, \tag{2.17}$$

where

$$B = \left(\begin{pmatrix} 4 & 1 \\ 0 & 4 \end{pmatrix}, e^{(2\lambda+1)\pi i/4} \right), \qquad A_\nu = \begin{pmatrix} 1 & 0 \\ 4\nu & 1 \end{pmatrix}.$$

It is normalized such that $\mathrm{pr}^2 = \mathrm{pr}$. Although the fact that pr is the projection to the plus-space is only proved for holomorphic modular forms in Kohnen's paper, his argument follows *mutatis mutandis* for non-holomorphic forms.

We define the Poincaré series of weight k and index m for $\Gamma_0(4)$ by

$$F_m^+(z) = \frac{3}{2} F_m(z, k/2) \mid_k \mathrm{pr}, \tag{2.18}$$

where $F_m(z, k/2)$ is the special value at $s = k/2$ of the weight k series defined in (2.14).

Theorem 2.8. *Assume that m is a negative integer and $(-1)^\lambda m \equiv 0, 1 \pmod 4$. The Poincaré series $F_m^+(z)$ is a weak Maass form of weight k for the group $\Gamma_0(4)$ satisfying the plus-space condition.*

1. *If $k > 3/2$, then $F_m^+(z) \in M^!_{\lambda+\frac{1}{2}}$, and it has a Fourier expansion of the form*

$$F_m^+(z) = q^m + \sum_{\substack{n>0 \\ (-1)^\lambda n \equiv 0,1 \pmod 4}} c^+(n) q^n,$$

   where

$$\begin{aligned} c^+(n) = (-1)^{\lfloor(\lambda+1)/2\rfloor}(1-(-1)^\lambda i)\pi\sqrt{2}\left|\frac{n}{m}\right|^{\lambda/2-1/4} \\ \times \sum_{\substack{c>0 \\ c\equiv 0\ (4)}} (1+\delta_{\mathrm{odd}}(c/4))\frac{K_k(m,n,c)}{c} I_{\lambda-1/2}\left(\frac{4\pi}{c}\sqrt{|mn|}\right), \end{aligned} \tag{2.19}$$

   for $n > 0$. At the other cusps, $F_m^+(z)$ is holomorphic (and actually vanishes).

2. *If $k = 3/2$, then the Fourier coefficients $c^+(n)$ with positive index n are still given by (2.19). Near the cusp at ∞, the function $F_m^+(z) - e(mz)$ decays as y^{1-k}. Near the other cusps the function $F_m^+(z)$ decays as y^{1-k}.*

Proof. Using Theorem 2.6, the projection of $F_m(z, k/2)$ to the plus-space can be calculated in exactly the same way as the projection of the usual holomorphic Poincaré series (see Proposition 4 of [20]). □

To obtain the Fourier expansion completely for $k = 3/2$, we need to compute $\xi_k(F_m^+)$. This can be done most easily by comparing the non-holomorphic part of F_m^+ with the non-holomorphic part of Zagier's Eisenstein series $G(z)$ of weight 3/2 (see [26]). This Eisenstein series can be constructed by taking the special value at $s = k/2$ of the

Eisenstein series $E(z,s) = F(z, y^{s-k/2})$ of weight $3/2$ for $\Gamma_0(4)$, and by computing its projection to the plus-space. Thus, we have that

$$G(z) = \frac{3}{2} E(z, k/2) \mid_k \mathrm{pr}\,.$$

The Fourier expansion of $G(z)$ was determined by Zagier, and it is given by

$$G(z) = \sum_{n=0}^{\infty} H(n) q^n + \frac{1}{16\pi\sqrt{y}} \sum_{n=-\infty}^{\infty} \beta(4\pi n^2 y) q^{-n^2}, \tag{2.20}$$

where $H(0) = \zeta(-1) = -\frac{1}{12}$, and $\beta(s) = \int_1^\infty t^{-3/2} e^{-st} dt$.

Proposition 2.9. *If $k = 3/2$ and m is a negative integer with $-m \equiv 0, 1 \pmod 4$, then the following are true.*

1. *If $-m$ is the square of a non-zero integer, then*

$$F_m^+(z) + 24G(z) \in M_{3/2}^!.$$

2. *If $-m$ is not a square, then $F_m^+(z) \in M_{3/2}^!$.*

Proof. Clearly, the assertion of Proposition 2.7 also holds for F_m^+. The function $\xi_{3/2}(F_m^+)$ is a multiple of θ. On the other hand, direct computation reveals that

$$\xi_{3/2}(G(z)) = -\frac{1}{16\pi} \theta(z).$$

Hence there is a constant r such that $f = F_m^+ + rG$ is annihilated by $\xi_{3/2}$. For this choice of r, we have that $f \in M_{3/2}^!$.

To determine r, we use the remark at the end of §5 of [27]. As a consequence of the residue theorem on compact Riemann surfaces, the constant term in the q-expansion of fg has to vanish for any $g \in M_{1/2}^!$. We apply this for $g = \theta$. Since $f = q^m + rH(0) + O(q)$, we obtain the assertion. $\square$

Using Proposition 2.9, we can then prove Theorem 2.3, giving the formula

$$B_1(D,n) = 24\delta_{\square,D}H(n) - (1+i)\sum_{\substack{c>0\\ c\equiv 0\ (4)}} (1+\delta_{\mathrm{odd}}(c/4))\frac{K_{3/2}(-D,n,c)}{\sqrt{cD}}\ \sinh\left(\frac{4\pi}{c}\sqrt{Dn}\right)$$

for the coefficients of each $g_D(z)$.

Proof of Theorem 2.3. In view of Proposition 2.9, we obviously have that

$$g_D(z) = F^+_{-D}(z) + 24\delta_{\square,D}G(z).$$

The assertion follows from Theorem 2.8, and the fact that $I_{1/2}(z) = \left(\frac{2}{\pi z}\right)^{1/2}\sinh(z)$ (see [23] formula (10.2.13)). $\square$

Chapter 3

A New Proof of Zagier's Modularity Theorem

In this chapter we give a new proof of Zagier's modularity theorem (Theorem 1.3), showing that generalized traces $\mathrm{Tr}_m(d)$ of singular moduli occur as the negatives of the coefficients $B_m(1,d)$ of the weight 3/2 modular form g_1, hit with the Hecke operator $T(m^2)$. Instead of the argument Zagier made using recurrence relations satisfied by the traces and by the coefficients, this proof uses Kloosterman sums and the theory of weakly holomorphic Poincaré series to directly show equality. More specifically, we relate formulas given by Duke for the $\mathrm{Tr}_m(d)$ to the coefficients of certain Poincaré series computed by the author, Bruinier, and Ono that appear in computing the $B_m(1,d)$. The ideas here first appeared in [17].

Remark. This proof generalizes the proof of the $m = 1$ case given by Duke in [10].

3.1 Preliminaries

We want to give a new proof that the mth trace can be written as a coefficient of a modular form; specifically, we want to show that $\mathrm{Tr}_m(d) = -B_m(1,d)$. Duke [10] gives a "Kloosterman sum" proof of the modularity of the traces for the $m = 1$ case by adapting a method of Tóth [24]. Here we generalize this proof to all integers $m \geq 1$.

In the formulas that follow, $\sum_{c\equiv 0(4)}$ will denote a sum over positive integers c divisible by 4. In addition, hereafter we will write the sum over all residue classes $x (\mathrm{mod}\ c)$ as $\sum_{x(c)}$.

We begin by recalling Theorem 2.3, in which we used the theory of weakly holomorphic Poincaré series to prove the following formulas for the coefficients $B_1(D,d)$ of the modular forms g_D.

Theorem 3.1. *Let D be a positive integer with $D \equiv 0, 1 \pmod 4$. Then the Fourier coefficient $B_1(D,d)$ with positive index d, where $d \equiv 0, 3 \pmod 4$, is given by*

$$B_1(D,d) = 24\delta_{\square,D}H(d) - (1+i)\sum_{c\equiv 0\ (4)} \left(1+\delta_{\mathrm{odd}}\left(\frac{c}{4}\right)\right) \frac{K_{3/2}(-D,d;c)}{\sqrt{cD}} \sinh\left(\frac{4\pi}{c}\sqrt{Dd}\right).$$

Here $\delta_{\square,D} = 1$ if n is a square, and $\delta_{\square,D} = 0$ otherwise.

Duke proves the following formula for the $\mathrm{Tr}_m(d)$.

Theorem 3.2 ([10], Proposition 4). *For any positive integer m and discriminant $-d$,*

$$\mathrm{Tr}_m(d) = -24H(d)\sigma(m) + \sum_{c\equiv 0(4)} S_d(m,c) \sinh\left(\frac{4\pi m\sqrt{d}}{c}\right),$$

where

$$S_d(m,c) = \sum_{x^2\equiv -d(c)} e\left(\frac{2mx}{c}\right).$$

We combine these two formulas to obtain a new proof of Theorem 1.3. We require the following lemma.

Lemma 3.3. *For $d \equiv 0, 3 \pmod 4$ and $k, n, c \geq 1$ with $nc \equiv 0 \pmod 4$ and $(c,k) = 1$, we have*

$$S_d(nk, nc) = \sum_{\substack{h|n \\ hc\equiv 0(4)}} (1+i)\left(1+\delta_{\mathrm{odd}}\left(\frac{hc}{4}\right)\right)\frac{1}{\sqrt{hc}}K_{3/2}\left(-h^2k^2, d; hc\right).$$

Remark. This lemma can be obtained by slightly modifying work of Kohnen [20]; we give instead a more direct and elementary proof depending only on classical facts about Gauss sums.

As a preliminary to our proof of Lemma 3.3, we define the Gauss sum

$$G(a,b;c) = \sum_{x(c)} e\left(\frac{ax^2+bx}{c}\right). \tag{3.1}$$

Note that $G(a,b;c)$ vanishes if $(a,c)>1$ unless $(a,c)|b$, in which case

$$G(a,b;c) = (a,c)G\left(\frac{a}{(a,c)}, \frac{b}{(a,c)}; \frac{c}{(a,c)}\right). \tag{3.2}$$

For $(a,c)=1$, we can evaluate ([2], Theorems 1.5.1, 1.5.2 and 1.5.4)

$$G(a,0;c) = \begin{cases} 0 & \text{if } 0<c\equiv 2 \pmod 4, \\ \varepsilon_c\sqrt{c}\left(\frac{a}{c}\right) & \text{if } c \text{ odd}, \\ (1+i)\varepsilon_a^{-1}\sqrt{c}\left(\frac{c}{a}\right) & \text{if } a \text{ odd and } 4|c. \end{cases} \tag{3.3}$$

We will also need the following basic lemma.

Lemma 3.4. *For integers a,b,c with b odd and $(a,4c)=1$, we have $G(a,b;4c)=0$.*

Proof. Replacing x with $x+2c$ in the sum defining $G(a,b;4c)$ simply rearranges the sum. But this change of variables also introduces a factor $e(b/2)=-1$, so we must have $G(a,b;4c)=0$. □

Proof of Lemma 3.3. Assume that $d\equiv 0,3 \pmod 4$ and $k,n,c\geq 1$ with $nc\equiv 0 \pmod 4$ and $(c,k)=1$.

Multiplied by nc, the right side of the equation we are trying to establish is

$$\sum_{\substack{h|n \\ hc\equiv 0(4)}} (1+i)\frac{n\sqrt{c}}{\sqrt{h}}\left(1+\delta_{\text{odd}}\left(\frac{hc}{4}\right)\right) K_{3/2}\left(-h^2k^2, d; hc\right)$$

$$= \sum_{\substack{h|n \\ hc\equiv 0(4)}} (1+i)\frac{n\sqrt{c}}{\sqrt{h}} K_{3/2}\left(-h^2k^2, d; hc\right) + \sum_{\substack{h|n \\ \frac{hc}{4}\text{ odd}}} (1+i)\frac{n\sqrt{c}}{\sqrt{h}} K_{3/2}\left(-h^2k^2, d; hc\right).$$

Write this as $S_0 + S_1$ for brevity. For $hc/4$ odd, we can rewrite the Kloosterman sum $K_{3/2}(-h^2k^2, d; hc)$ as ([15], Lemma 2)

$$K_{3/2}\left(-h^2k^2, d; hc\right) = \left(\cos\frac{\pi(d-h^2k^2)}{2} - \sin\frac{\pi(d-h^2k^2)}{2}\right)(1-i)\varepsilon_{\frac{hc}{4}} S\left(-\overline{4}h^2k^2, \overline{4}d; \frac{hc}{4}\right), \tag{3.4}$$

where $S(m,n;q) = \sum_{a(q)} \left(\frac{a}{q}\right) e\left(\frac{ma+n\overline{a}}{q}\right)$ is a Salié sum. Since $d \equiv 0, 3 \pmod 4$, the cosine-sine term is 1 unless $d \equiv 3 \pmod 4$ and hk is odd, in which case it is -1. We thus have

$$S_1 = \sum_{\substack{h|n \\ \frac{hc}{4}\text{ odd}}} \frac{2n\sqrt{c}}{\sqrt{h}}\left(e\left(\frac{d}{2}\right)\right)^{hk} \varepsilon_{\frac{hc}{4}} S\left(-\overline{4}h^2k^2, \overline{4}d; \frac{hc}{4}\right).$$

On the other hand, we have by definition that

$$ncS_d(nk, nc) = nc\sum_{x^2\equiv -d(nc)} e\left(\frac{2kx}{c}\right) = \sum_{x(nc)} e\left(\frac{2kx}{c}\right)\sum_{a(nc)} e\left(\frac{a(x^2+d)}{nc}\right)$$

$$= \sum_{a(nc)} e\left(\frac{ad}{nc}\right) G(a, 2nk; nc). \tag{3.5}$$

The properties of G in (3.2) allow this to be rewritten as

$$\sum_{A|(2nk,nc)} \sum_{(a,nc)=A} e\left(\frac{ad}{nc}\right) A\cdot G\left(\frac{a}{A}, \frac{2nk}{A}; \frac{nc}{A}\right) = \sum_{A|(2nk,nc)} \sum_{(b,\frac{nc}{A})=1} e\left(\frac{bd}{nc/A}\right) A\cdot G\left(b, \frac{2nk}{A}; \frac{nc}{A}\right).$$

Since $(c,k) = 1$, we have $(2nk, nc) = n$ if c is odd, and $2n$ if c is even.

We treat first the case in which c is odd. We have

$$ncS_d(nk,nc) = \sum_{A|n} \sum_{(b,\frac{nc}{A})=1} e\left(\frac{bd}{nc/A}\right) A \cdot G\left(b, \frac{2nk}{A}; \frac{nc}{A}\right)$$
$$= \sum_{h|n} \sum_{(b,hc)=1} e\left(\frac{bd}{hc}\right) \frac{n}{h} \cdot G(b, 2hk, hc).$$

We complete the square in the definition of $G(b, 2hk, hc)$ to see that this equals

$$\sum_{h|n} \sum_{(b,hc)=1} e\left(\frac{bd - \bar{b}h^2k^2}{hc}\right) \frac{n}{h} \cdot G(b, 0, hc),$$

and apply (3.3) to get

$$ncS_d(nk,nc) = \sum_{\substack{h|n \\ 4|hc}} \sum_{(b,hc)=1} e\left(\frac{bd - \bar{b}h^2k^2}{hc}\right) (1+i)\varepsilon_b^{-1} \frac{n\sqrt{c}}{\sqrt{h}} \left(\frac{hc}{b}\right)$$
$$+ \sum_{\substack{h'|n \\ h'c\,\mathrm{odd}}} \sum_{(b,h'c)=1} e\left(\frac{bd - \bar{b}h'^2k^2}{h'c}\right) \varepsilon_{h'c} \frac{n\sqrt{c}}{\sqrt{h'}} \left(\frac{b}{h'c}\right).$$

Writing $4h' = h$ in the second term, we get

$$\sum_{\substack{h|n \\ 4|hc}} (1+i)\frac{n\sqrt{c}}{\sqrt{h}} K_{3/2}(-h^2k^2, d; hc) + \sum_{\substack{h|n \\ \frac{hc}{4}\,\mathrm{odd}}} \sum_{(b,\frac{hc}{4})=1} e\left(\frac{bd - \overline{16b}h^2k^2}{hc/4}\right) \varepsilon_{\frac{hc}{4}} \frac{2n\sqrt{c}}{\sqrt{h}} \left(\frac{b}{hc/4}\right).$$

Replacing b by $\bar{4}b$ shows that this precisely equals $S_0 + S_1$.

We now assume that $2|c$, so that k is odd. We have

$$ncS_d(nk,nc) = \sum_{A|2n} \sum_{(b,\frac{nc}{A})=1} e\left(\frac{bd}{nc/A}\right) A \cdot G\left(b, \frac{2nk}{A}; \frac{nc}{A}\right).$$

If $A|nk$, then $A|n$ since we also have $A|nc$; this lets us split the sum as

$$\sum_{A|2n} = \sum_{A|n} + \sum_{\substack{A|2n \\ \frac{2nk}{A}\,\mathrm{odd}}}$$

and obtain

$$ncS_d(nk,nc) = S_0 + \sum_{\substack{h'|2n \\ h'k\,\mathrm{odd}}} \sum_{(b,\frac{h'c}{2})=1} e\left(\frac{bd}{h'c/2}\right)\frac{2n}{h'} \cdot G\left(b, h'k; \frac{h'c}{2}\right),$$

where S_0 is obtained from the first term as in the previous case.

In the case that $c/2$ is odd, we can complete the square and write

$$G\left(b, h'k; \frac{h'c}{2}\right) = e\left(\frac{-\overline{4b}h'^2k^2}{h'c/2}\right) G\left(b, 0; \frac{h'c}{2}\right).$$

We again apply (3.3) to get

$$ncS_d(nk,nc) = S_0 + \sum_{\substack{h'|2n \\ h'k\,\mathrm{odd}}} \sum_{(b,\frac{h'c}{2})=1} e\left(\frac{bd - \overline{4b}h'^2k^2}{h'c/2}\right)\varepsilon_{\frac{h'c}{2}}\frac{n\sqrt{2c}}{\sqrt{h'}}\left(\frac{b}{h'c/2}\right).$$

Writing $2h' = h$ and replacing b by $\overline{4}b$ shows that this again equals $S_0 + S_1$.

If $8|c$, then $S_1 = 0$, and Lemma 3.4 shows that $ncS_d(nk,nc) = S_0$.

The remaining case is $c = 4w$, for w odd. In this case, we have

$$S_1 = \sum_{\substack{h|n \\ h\,\mathrm{odd}}} \frac{4n\sqrt{w}}{\sqrt{h}} e\left(\frac{d}{2}\right)\varepsilon_{hw}\sum_{(b,hw)=1}\left(\frac{b}{hw}\right)e\left(\frac{\overline{4}bd - \overline{4b}h^2k^2}{hw}\right),$$

$$ncS_d(nk,nc) = S_0 + \sum_{\substack{h|n \\ h\,\mathrm{odd}}} \sum_{(b,2hw)=1} e\left(\frac{bd}{2hw}\right)\frac{2n}{h}\sum_{x(2hw)} e\left(\frac{bx^2 + hkx}{2hw}\right).$$

But each b coprime to $2hw$ is equal (mod $2hw$) to $hw - 2z$ for some z coprime to hw, so we have

$$ncS_d(nk,nc) = S_0 + \sum_{\substack{h|n \\ h\,\mathrm{odd}}} \sum_{(b,hw)=1} e\left(\frac{(hw-2b)d}{2hw}\right)\frac{2n}{h}\sum_{x(2hw)} e\left(\frac{(hw-2b)x^2 + hkx}{2hw}\right).$$

Simplifying, and noting that $e\left(\frac{x^2}{2}\right) = e\left(\frac{x}{2}\right)$ for all integers x, we have

$$\begin{aligned} ncS_d(nk, nc) &= S_0 + \sum_{\substack{h|n \\ h\,\text{odd}}} \sum_{(b,hw)=1} e\left(\frac{d}{2}\right) e\left(\frac{-bd}{hw}\right) \frac{2n}{h} G(-2b, hk + hw; 2hw) \\ &= S_0 + \sum_{\substack{h|n \\ h\,\text{odd}}} \sum_{(b,hw)=1} e\left(\frac{d}{2}\right) e\left(\frac{-bd}{hw}\right) \frac{4n}{h} G(-b, \overline{2}hk; hw). \end{aligned}$$

Since hw is odd, we complete the square to get

$$G(-b, \overline{2}hk; hw) = e\left(\frac{\overline{16b}h^2k^2}{hw}\right) G(-b, 0; hw) = e\left(\frac{\overline{16b}h^2k^2}{hw}\right) \varepsilon_{hw} \sqrt{hw} \left(\frac{-b}{hw}\right).$$

Replacing b by $-\overline{4}b$ gives $ncS_d(nk, nc) = S_0 + S_1$, proving the lemma. $\square$

3.2 Proof of Zagier's modularity theorem

We now prove Theorem 1.3.

Proof of Theorem 1.3. From Zagier's duality theorem (Theorem 1.2) and equation (1.6), we know that $-B_m(1, d) = -\sum_{r|m} rB(r^2, d)$. Applying Theorem 3.1 with $D = r^2$, we find that

$$\begin{aligned} -B_m(1, d) = &-24H(d)\sigma(m)+ \\ &\sum_{r|m} \sum_{n|r} \sum_{\substack{\frac{rc}{n} \equiv 0(4) \\ (c,n)=1}} (1 + i) \left(1 + \delta_{\text{odd}}\left(\frac{rc}{4n}\right)\right) \frac{K_{3/2}(-r^2, d; \frac{rc}{n})}{\sqrt{\frac{rc}{n}}} \sinh\left(\frac{4\pi n\sqrt{d}}{c}\right). \end{aligned}$$

Writing $r = hn$, we get $\sum_{r|m} \sum_{n|r} = \sum_{n|m} \sum_{h|\frac{m}{n}}$. Replace n by m/n inside the sum on n, and write $\sum_{h|n} \sum_{hc\equiv 0(4), (c,\frac{m}{n})=1} = \sum_{nc\equiv 0(4), (c,\frac{m}{n})=1} \sum_{h|n, hc\equiv 0(4)}$ to get

$$\begin{aligned} -B_m(1, d) = &-24H(d)\sigma(m)+ \\ &\sum_{n|m} \sum_{\substack{nc\equiv 0(4) \\ (c,\frac{m}{n})=1}} \sum_{\substack{h|n \\ hc\equiv 0(4)}} (1 + i) \left(1 + \delta_{\text{odd}}\left(\frac{hc}{4}\right)\right) \frac{K_{3/2}(\frac{-h^2m^2}{n^2}, d; hc)}{\sqrt{hc}} \sinh\left(\frac{4\pi m\sqrt{d}}{nc}\right). \end{aligned}$$

We rewrite Theorem 3.2 as

$$\mathrm{Tr}_m(d) = -24H(d)\sigma(m) + \sum_{n|m} \sum_{\substack{nc\equiv 0(4)\\ (c,\frac{m}{n})=1}} S_d(m, nc)\sinh\left(\frac{4\pi m\sqrt{d}}{nc}\right). \tag{3.6}$$

Letting $m = nk$ in Lemma 3.3 shows that the coefficients of the hyperbolic sine terms in the two expressions are equal, and we have $\mathrm{Tr}_m(d) = -B_m(1, d)$, proving the theorem. $\square$

Chapter 4

p-Divisibility of Traces of Singular Moduli

4.1 Examples

To begin, we examine the p-divisibility of $\mathrm{Tr}(p^n d)$ for various values of p, n, and d.

$$\mathrm{Tr}(3^2 \cdot 11) = -37616060991672 \equiv 0 \pmod{3}$$

$$\mathrm{Tr}(3^4 \cdot 11) = 5322552412520219020968608933614304 5073448 \equiv 0(3^2)$$

$$\mathrm{Tr}(3^6 \cdot 11) \equiv 0 \pmod{3^3}$$

$$\mathrm{Tr}(3^2 \cdot 20) = 2018504138610379536 \equiv 0 \pmod{3}$$

$$\mathrm{Tr}(3^4 \cdot 20) = -82241104354244343668036845539817349528284508241030106 08$$

$$\equiv 0 \pmod{3^2}$$

$$\mathrm{Tr}(3^6 \cdot 20) \equiv 0 \pmod{3^3}$$

$$\mathrm{Tr}(5^2 \cdot 11) = -4223010805195936842 1000 \equiv 0 \pmod{5}$$

$$\mathrm{Tr}(5^4 \cdot 11) \equiv 0 \pmod{5^2}$$

Such congruences continue to appear if more examples are computed. In this chapter, we examine such phenomena. The results here first appeared in [16].

4.2 Results

Suppose that p is an odd prime and that s is a positive integer. When p is inert or ramified in particular quadratic number fields, Ahlgren and Ono [1] proved many congruences for traces of singular moduli modulo p^s. In addition, they gave an elementary argument that $\mathrm{Tr}(p^2d) \equiv 0 \pmod{p}$ when p splits in $\mathbb{Q}(\sqrt{-d})$. In a recent preprint, Edixhoven [12] extended their observation and proved that if $\left(\frac{-d}{p}\right) = 1$, then

$$\mathrm{Tr}(p^{2n}d) \equiv 0 \pmod{p^n}. \tag{4.1}$$

In another recent preprint, Boylan [4] exactly computes $\mathrm{Tr}(2^{2n}d)$, giving an analogous result when $p = 2$.

Remark. The aim of Edixhoven's paper is to show that the p-adic geometry of modular curves can be used to study p-adic properties of traces of $f \in \mathbb{Z}[j]$, of which Zagier's trace is one. The congruences we cite are a special case of his more general result.

Remark. Treneer [25] has extended Ahlgren and Ono's work on the inert and ramified cases to describe congruences for certain coefficients of weakly holomorphic modular forms in much greater generality.

In this chapter we obtain an exact formula for the coefficient $B_1(D, p^{2n}d)$ of $q^{p^{2n}d}$ in g_D, allowing us to obtain Edixhoven's congruences as a corollary. Specifically, we prove the following theorem.

Theorem 4.1. *If p is a prime and d, D, n are positive integers such that $-d, D \equiv 0, 1 \pmod 4$, then*

$$B_1(D, p^{2n}d) = p^n B_1(p^{2n}D, d) + \sum_{k=0}^{n-1} \left(\frac{D}{p}\right)^{n-k-1} \left(B_1\left(\frac{D}{p^2}, p^{2k}d\right) - p^{k+1} B_1\left(p^{2k}D, \frac{d}{p^2}\right)\right)$$
$$+ \sum_{k=0}^{n-1} \left(\frac{D}{p}\right)^{n-k-1} \left(\left(\left(\frac{D}{p}\right) - \left(\frac{-d}{p}\right)\right) p^k B_1(p^{2k}D, d)\right).$$

The following corollaries follow immediately.

Corollary 4.2. *For a prime p and positive integers d, D, n such that $-d, D \equiv 0, 1 \pmod 4$, if $\left(\frac{D}{p}\right) = \left(\frac{-d}{p}\right) \neq 0$, then $B_1(D, p^{2n}d) = p^n B_1(p^{2n}D, d)$.*

Corollary 4.3. *If $\left(\frac{-d}{p}\right) = 1$, then $\mathrm{Tr}(p^{2n}d) = -p^n B_1(p^{2n}, d)$.*

Remark. These results can easily be extended to Zagier's generalized traces $\mathrm{Tr}_m(d)$.

Remark. As the examples at the beginning of this chapter show, for the primes $3, 5, 7$, and 11, these congruences are not optimal; for example, it appears that if $\left(\frac{-d}{p}\right) = 1$, then $\mathrm{Tr}(11^2 d) \equiv 0 \pmod{11^2}$. In these cases, it should be possible to obtain more exact results, similar to Boylan's work for $p = 2$. Along these lines, Guerzhoy [13] recently proved that if $p \leq 11$ and $\left(\frac{-d}{p}\right) = \left(\frac{D}{p}\right) \neq 0$, then the p-adic limit

$$\lim_{n\to\infty} p^{-n}(-B_1(D, p^{2n}d)) = 0,$$

uniformly proving higher congruences without providing information about specific primes.

4.3 Proving new coefficient relationships

To give the proof of Theorem 4.1, we need the following formulas for the action of the Hecke operators, given in Chapter 1 as equations (1.4) and (1.5). For a prime p, we have

$$A_p(D, d) = pA_1(p^2D, d) + \left(\frac{D}{p}\right) A_1(D, d) + A_1\left(\frac{D}{p^2}, d\right), \tag{4.2}$$

$$B_p(D,d) = B_1(D,p^2d) + \left(\frac{-d}{p}\right)B_1(D,d) + pB_1\left(D,\frac{d}{p^2}\right). \tag{4.3}$$

Note that these formulas hold as well for $p = 2$, although this case was not included in [16].

Applying Zagier's duality theorem (Theorem 1.2) to equation (1.4), we get

$$B_p(D,d) = pB_1(p^2D,d) + \left(\frac{D}{p}\right)B_1(D,d) + B_1\left(\frac{D}{p^2},d\right).$$

Apply equation (1.5) to get

$$B_1(D,p^2d) = pB_1(p^2D,d)+\left(\frac{D}{p}\right)B_1(D,d)+B_1\left(\frac{D}{p^2},d\right)-\left(\frac{-d}{p}\right)B_1(D,d)-pB_1\left(D,\frac{d}{p^2}\right). \tag{4.4}$$

These observations alone suffice to prove Theorem 4.1.

Proof of Theorem 4.1. We prove the theorem by induction on n. The $n = 1$ case is just equation (4.4). For $n > 1$, assume the theorem holds up to $n - 1$.

Replacing d by $p^{2n-2}d$ in equation (4.4) gives

$$B_1(D,p^{2n}d) = p(B_1(p^2D,p^{2n-2}d)-B_1(D,p^{2n-4}d))+B_1\left(\frac{D}{p^2},p^{2n-2}d\right)+\left(\frac{D}{p}\right)B_1(D,p^{2n-2}d). \tag{4.5}$$

Note that for $1 \le k \le n-2$, replacing D with $p^{2k}D$ and d with $p^{2n-2k-2}d$ in (4.4) gives

$$\begin{aligned} &p^k(B_1(p^{2k}D,p^{2n-2k}d) - B_1(p^{2k-2}D,p^{2n-2k-2}d)) \\ =&p^{k+1}(B_1(p^{2k+2}D,p^{2n-2k-2}d) - B_1(p^{2k}D,p^{2n-2k-4}d)). \end{aligned} \tag{4.6}$$

Making a similar replacement in (4.4) with $k = n-1$, we get

$$\begin{aligned} &p^{n-1}(B_1(p^{2n-2}D,p^2d) - B_1(p^{2n-4}D,d)) \\ =&p^nB_1(p^{2n}D,d) - p^nB_1\left(p^{2n-2}D,\frac{d}{p^2}\right) - p^{n-1}\left(\frac{-d}{p}\right)B_1(p^{2n-2}D,d). \end{aligned} \tag{4.7}$$

Using (4.6) $n-2$ times and (4.7), we obtain

$$\begin{aligned} &p(B_1(p^2D,p^{2n-2}d)-B_1(D,p^{2n-4}d)) \\ =&p^nB_1(p^{2n}D,d)-p^nB_1\left(p^{2n-2}D,\frac{d}{p^2}\right)-p^{n-1}\left(\frac{-d}{p}\right)B_1(p^{2n-2}D,d). \end{aligned} \tag{4.8}$$

Substituting (4.8) in (4.5) gives

$$\begin{aligned} B_1(D,p^{2n}d)=&p^nB_1(p^{2n}D,d)-p^nB_1\left(p^{2n-2}D,\frac{d}{p^2}\right)-p^{n-1}\left(\frac{-d}{p}\right)B_1(p^{2n-2}D,d) \\ &+B_1\left(\frac{D}{p^2},p^{2n-2}d\right)+\left(\frac{D}{p}\right)B_1(D,p^{2n-2}d). \end{aligned}$$

Apply the induction hypothesis to expand the last term; after simplifying, the theorem follows. □

Chapter 5

Twisted Traces

In this chapter we discuss twisted traces and their representation as heretofore unused coefficients of the half integral weight modular forms f_d. We obtain a new expression for computing twisted traces as an infinite series, and obtain some p-divisibility properties and consequent congruences.

5.1 Definitions

The traces we have discussed may be generalized by adding a twist as follows. For a fundamental discriminant D, we define the genus character χ_D to be the character assigning a quadratic form $Q = (a, b, c)$, of discriminant divisible by D, the value

$$\chi_D(Q) = \begin{cases} 0 & \text{if } (a, b, c, D) > 1, \\ \left(\frac{D}{n}\right) & \text{if } (a, b, c, D) = 1, \end{cases}$$

where n is any integer represented by Q and coprime to D. This is independent of the choice of n, and for a form Q of discriminant $-dD$, we have $\chi_D = \chi_{-d}$ if $-d$ and D are both fundamental discriminants.

Following Zagier, for fundamental discriminants $D > 0$ we then define a "twisted trace" $\mathrm{Tr}_m(D, d)$ as

$$\mathrm{Tr}_m(D, d) = \sum_{Q \in \mathcal{Q}_{dD}/\Gamma} \frac{\chi_D(Q) j_m(\alpha_Q)}{\omega_Q}. \tag{5.1}$$

We have discussed the interpretation of the Fourier coefficients $A_m(D,d)$ of the weight $1/2$ forms f_d as traces when D is equal to 1, or, more generally, when D is a square. If D is not a square, then, as observed by Zagier, the other coefficients of the f_d may be interpreted in terms of twisted traces in the following manner.

Theorem 5.1 ([27], Theorem 6). *If $m \geq 1, -d < 0$ is a discriminant, and $D > 0$ is a fundamental discriminant, then*

$$\mathrm{Tr}_m(D,d) = A_m(D,d)\sqrt{D}.$$

5.2 New expressions for twisted traces

Our next result is a new expression for twisted traces as an infinite series, which is a generalization of Duke's expression in Theorem 3.2 for the traces $\mathrm{Tr}(d)$ of $j(z) - 744$.

Theorem 5.2. *If $D, -d \equiv 0, 1 \pmod 4$ are a positive fundamental discriminant and a negative discriminant, respectively, with $D > 1$, and $m \geq 1$ is an integer, we have*

$$\mathrm{Tr}_m(D,d) = \sum_{\substack{c \equiv 0(4) \\ c > 0}} S_{D,d}(m,c) \sinh\left(\frac{4\pi m\sqrt{dD}}{c}\right),$$

where

$$S_{D,d}(m,c) = \sum_{\substack{x(c) \\ x^2 \equiv -Dd(c)}} \chi_D\left(\frac{c}{4}, x, \frac{x^2+Dd}{c}\right) e\left(\frac{2mx}{c}\right).$$

Proof of Theorem 5.2. We know from Zagier's work that

$$\mathrm{Tr}_m(D,d) = A_m(D,d)\sqrt{D}. \tag{5.2}$$

We combine this with equation (1.7) for fundamental discriminants,

$$A_m(D,d) = \sum_{n|m} n\left(\frac{D}{\frac{m}{n}}\right) A_1(n^2D, d),$$

and use the fact that $A_m(D,d) = -B_m(D,d)$ to see that

$$\mathrm{Tr}_m(D,d) = -\sum_{n|m} \sqrt{D} n \left(\frac{D}{\frac{m}{n}}\right) B(n^2 D, d).$$

We now apply Theorem 3.1 to write $B(n^2D,d)$ as an infinite sum, and obtain

$$\mathrm{Tr}_m(D,d) = \sum_{n|m} \sum_{c\equiv 0(4)} (1+i)\left(\frac{D}{\frac{m}{n}}\right) \left(1+\delta_{\mathrm{odd}}\left(\frac{c}{4}\right)\right) \frac{1}{\sqrt{c}} K_{3/2}(-n^2D, d; c) \sinh\left(\frac{4\pi n\sqrt{dD}}{c}\right).$$

Rewriting $\sum_{c\equiv 0(4)}$ as $\sum_{r|n}\sum_{rc\equiv 0(4),(c,n/r)=1}$ and changing the order of summation as in the previous section, this becomes

$$\sum_{c\equiv 0(4)} \sum_{k|(m,\frac{c}{4})} (1+i)\left(\frac{D}{k}\right) \left(1+\delta_{\mathrm{odd}}\left(\frac{c}{4k}\right)\right) \sqrt{\frac{k}{c}} K_{3/2}\left(\frac{-m^2D}{k^2}, d; \frac{c}{k}\right) \sinh\left(\frac{4\pi m\sqrt{Dd}}{c}\right).$$

We use the following identity, proved by Kohnen ([20], Proposition 5): For integers $a, m, d \geq 1$ and fundamental discriminants D satisfying $D, (-1)^\lambda d \equiv 0, 1 \pmod 4$, we have

$$\sum_{\substack{b(4a)\\ b^2\equiv(-1)^\lambda Dd(4a)}} \chi_D\left(a, b, \frac{b^2-(-1)^\lambda Dd}{4a}\right) e\left(\frac{mb}{2a}\right) = \tag{5.3}$$

$$\sum_{k|(m,a)} (1-(-1)^\lambda i)\left(\frac{D}{k}\right) \left(1+\delta_{\mathrm{odd}}\left(\frac{a}{k}\right)\right) \frac{\sqrt{k}}{2\sqrt{a}} K_{\lambda+1/2}\left(\frac{(-1)^\lambda m^2 D}{k^2}, d; \frac{4a}{k}\right).$$

This identity is stated only for $(-1)^\lambda dD > 0$ in Kohnen's paper, but the proof is virtually unchanged for the more general case.

Applying the identity with $\lambda = 1$ and $a = c/4$, Theorem 5.2 follows immediately. $\square$

Example. Let $D = 5, d = 3, m = 1$. There are two forms of discriminant -15; they are $Q_1 = [1,1,4]$ and $Q_2 = [2,1,2]$, with $\chi_5(Q_1) = 1$ and $\chi_5(Q_2) = -1$. We have

$$j(\alpha_{Q_1}) = \frac{-191025 - 85995\sqrt{5}}{2},$$
$$j(\alpha_{Q_2}) = \frac{-191025 + 85995\sqrt{5}}{2},$$

and therefore $\mathrm{Tr}(D,d) = j(\alpha_{Q_1}) - j(\alpha_{Q_2}) = -85995\sqrt{5}$.

If we compute the first 10 terms of the sum in Theorem 5.2 and divide by $\sqrt{5}$, we get $-85996.573\cdots$. The first 100 terms give $-85995.909\cdots$, and the first 1000 terms give $-85994.9562\cdots$.

5.3 p-divisibility of twisted traces

Because of Zagier's duality theorem (Theorem 1.2) relating the coefficients $A_m(D,d)$ of the weight 1/2 forms f_d to the coefficients $B_m(D,d)$ of the weight 3/2 forms g_D, we may express these twisted traces as

$$\mathrm{Tr}_m(D,d) = -B_m(D,d)\sqrt{D}.$$

Applying Theorem 4.1 then gives the following result describing the p-divisibility of the integral part of certain twisted traces of singular moduli.

Theorem 5.3. *If $D, -d \equiv 0, 1 \pmod 4$ are a positive fundamental discriminant and a negative discriminant and p is an odd prime with $\left(\frac{D}{p}\right) = \left(\frac{-d}{p}\right) \neq 0$, then*

$$\frac{-\mathrm{Tr}(D, p^{2n}d)}{\sqrt{D}} = p^n B_1(p^{2n}D, d).$$

This generalizes Edixhoven's congruences, telling us that if $\left(\frac{D}{p}\right) = \left(\frac{-d}{p}\right) \neq 0$, then the integral part of $\mathrm{Tr}(D, p^{2n}d)$ is divisible by p^n.

Chapter 6

Higher Levels and Different Weights

In this chapter we discuss generalizing Zagier duality and the interpretation of modular form coefficients as traces of special values of modular functions to modular forms of higher level or weights different from 1/2 or 3/2. We prove theorems about the p-divisibility of these generalized traces.

6.1 Higher levels

In Section 8 of [27], Zagier discusses replacing the function $j(z)$ with a modular function of higher level; specifically, he uses the Hauptmodul associated to other modular groups of genus zero. These ideas were explored further by Kim [19] and Osburn [21].

Let $\Gamma_0(N)^*$ be the group generated by $\Gamma_0(N)$ and all Atkin-Lehner involutions W_e for $e||N$. Here W_e may be represented by a matrix of the form $\frac{1}{\sqrt{e}}\begin{pmatrix} ex & y \\ Nz & ew \end{pmatrix}$ with $x, y, z, w \in \mathbb{Z}$ and $xwe - yzN/e = 1$. We may define j_N^* to be the corresponding Hauptmodul with Fourier expansion beginning $q^{-1} + 0 + \mathcal{O}(q)$.

For a positive integer d with $-d$ congruent to a square modulo $4N$, it is possible to define a trace $\mathrm{Tr}^{(N)}(d)$ by setting

$$\mathrm{Tr}^{(N)}(d) = \sum_Q \frac{j_N^*(\alpha_Q)}{\omega_Q},$$

where the sum is over $\Gamma_0^*(N)$-representatives of forms $Q = [a, b, c]$ with $a \equiv 0 \pmod{N}$. For every such d and N, there is a unique modular form $f_{d,N}$ in the space $M^!_{1/2}(\Gamma_0(4N))$ of weakly holomorphic modular forms of weight $1/2$ on $\Gamma_0(4N)$ with the Fourier expansion

$$f_{d,N}(z) = q^{-d} + \sum_{D>0} A^N(D, d)q^D.$$

There are only finitely many primes p such that $\Gamma_0(p)^*$ is of genus zero; specifically, if this is the case, then

$$p \in \{2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 41, 47, 59, 71\}.$$

For such p, Kim showed ([19], Corollary 3.6) that $\mathrm{Tr}^{(p)}(d) = A^p(1, d)$. Osburn exploited a Zagier-type duality between the coefficients of $f_{d,p}$ and the coefficients of corresponding weight $3/2$ forms $g^{(p)} \in M^!_{3/2}(\Gamma_0(4p))$ to prove that if ℓ is a prime with $\left(\frac{-d}{\ell}\right) = 1$, then

$$\mathrm{Tr}^{(p)}(\ell^2 d) \equiv 0 \pmod{\ell}.$$

Because the action of the Hecke operators on the modular forms $g^{(p)}$ and $f_{d,p}$ can be written in terms of the coefficients in exactly the same way as we saw in equations 1.4 and 1.5 when the Hecke operators acted on the f_d and g_D, the argument in the proof of Theorem 4.1 may be applied to obtain the following generalization of Osburn's theorem.

Theorem 6.1. *Let p be a prime so that $\Gamma_0(p)^*$ has genus zero, and let d be a positive integer such that $-d$ is congruent to a square modulo $4p$. If ℓ is a prime with $\left(\frac{-d}{\ell}\right) = 1$ and n is a positive integer, then*

$$\mathrm{Tr}^{(p)}(\ell^{2n} d) \equiv 0 \pmod{\ell^n}.$$

6.2 Other weights

Thus far we have seen duality between weakly holomorphic modular forms of weights 1/2 and 3/2. It is natural to ask whether similar dualities hold for other pairs of weights $(\lambda + \frac{1}{2}, \frac{3}{2} - \lambda)$. In the case where there are no nonzero cuspforms of weight 2λ on $\mathrm{SL}_2(\mathbb{Z})$, Zagier answered this question in the affirmative in section 9 of [27], showing dualities for $\lambda = 2, 3, 4, 5, 7$.

For $\lambda = 6$ and $\lambda \geq 8$, though, it is necessary to replace the modular forms of weight $\frac{3}{2} - \lambda$ with weak Maass forms of weight $\frac{3}{2} - \lambda$. Bringmann and Ono [5] revisit Zagier's results from the perspective of Maass-Poincaré series, extending the results of Bruinier, Ono, and the author in Chapter 2 and obtaining duality results for every $\lambda \in \mathbb{Z}$; in addition, they interpret all coefficients of the relevant weakly holomorphic modular forms as twisted traces of modular functions.

For any $\lambda \in \mathbb{Z}$, Bringmann and Ono define Poincaré series $F_\lambda(-m; z)$ for all $m \in \mathbb{Z}$ such that $(-1)^{\lambda+1} m \equiv 0, 1 \pmod 4$. For $\lambda \geq -6$ with $\lambda \neq -5$, it turns out that $F_\lambda(-m; z) \in M^!_{\lambda+\frac{1}{2}}$ and has a Fourier expansion of the form

$$F_\lambda(-m; z) = q^{-m} + \sum_{\substack{n \geq 0 \\ (-1)^\lambda n \equiv 0,1 \pmod 4}} b_\lambda(-m; n) q^n.$$

If $\lambda = -5$ or $\lambda \leq -7$, then $F_\lambda(-m; z)$ is a weak Maass form of weight $\lambda + \frac{1}{2}$, and its Fourier expansion is of the form

$$F_\lambda(-m; z) = B_\lambda(-m; z) + q^{-m} + \sum_{\substack{n \geq 0 \\ (-1)^\lambda n \equiv 0,1 \pmod 4}} b_\lambda(-m; n) q^n,$$

where $B_\lambda(-m; z)$ is a "non-holomorphic" part. They then establish the duality

$$b_\lambda(-m; n) = -b_{1-\lambda}(-n; m).$$

For $\lambda = 2, 3, 4, 5, 7$, this duality is between coefficients of weakly holomorphic modular forms on both sides, and since there are no cuspforms in $M^!_{\lambda+\frac{1}{2}}$ for these λ, we may again construct an argument similar to that in Theorem 4.1 to prove the following.

Theorem 6.2. *Suppose $\lambda \in \{2, 3, 4, 5, 7\}$. If p is prime and k, m, n are positive integers with $(-1)^{(\lambda+1)}m, (-1)^\lambda n \equiv 0, 1 \pmod 4$, then $b_\lambda(-m, p^{2k}n)$ is equal to*

$$\sum_{\ell=0}^{k-1} \left(\frac{(-1)^{\lambda+1}m}{p}\right)^{k-\ell-1} \left(\left(\frac{(-1)^{\lambda+1}m}{p}\right) - \left(\frac{(-1)^\lambda n}{p}\right)\right) p^{\lambda(k+\ell)-k} b_\lambda(-p^{2\ell}m, n)$$
$$+ \sum_{\ell=0}^{k-1} \left(\frac{(-1)^{\lambda+1}m}{p}\right)^{k-\ell-1} \left(p^{(\lambda-1)(k-\ell-1)} b_\lambda(\frac{-m}{p^2}, p^{2\ell}n) - p^{\lambda(k+\ell+1)-k} b_\lambda(-p^{2\ell}m, \frac{n}{p^2})\right)$$
$$+ p^{(2\lambda-1)k} b_\lambda(-p^{2k}m, n).$$

Corollary 6.3. *If $\left(\frac{(-1)^{\lambda+1}m}{p}\right) = \left(\frac{(-1)^\lambda n}{p}\right) \neq 0$, then*

$$b_\lambda(-m, p^{2k}n) = p^{2k\lambda-k} b_\lambda(-p^{2k}m, n).$$

Bringmann and Ono also interpret these coefficients in terms of traces of certain Poincaré series $\mathfrak{F}_\lambda$ first defined by Niebur. For a modular invariant f, they define

$$\mathrm{Tr}_{D_1}(f; D_2) = \sum_{Q \in \mathcal{Q}_{|D_1 D_1|}/\Gamma} \frac{\chi_{D_1}(Q) f(\alpha_Q)}{\omega_Q}.$$

They then establish that

$$b_\lambda(-m; n) = \frac{(-1)^{\lfloor(\lambda+1)/2\rfloor} n^{(\lambda-1)/2}}{m^{\lambda/2}} \cdot \mathrm{Tr}_{(-1)^{\lambda+1}m}(\mathfrak{F}_\lambda; n).$$

Applying Corollary 6.3, we find that if $\left(\frac{(-1)^{\lambda+1}m}{p}\right) = \left(\frac{(-1)^\lambda n}{p}\right) \neq 0$, then

$$\mathrm{Tr}_{(-1)^{\lambda+1}m}(\mathfrak{F}_\lambda; p^{2k}n) = p^{k\lambda}(-1)^{\lfloor(\lambda+1)/2\rfloor} b_\lambda(-p^{2k}m, n) \frac{m^{\lambda/2}}{n^{(\lambda-1)/2}},$$

again getting a congruence modulo $p^{k\lambda}$ similar to Edixhoven's.

Chapter 7

Divisibility criteria for class numbers

Recently, Guerzhoy proved (see Corollary 2.4 (a) of [14]) that if $p \in \{3, 5, 7, 13\}$ and $-d < -4$ is a fundamental discriminant, then one has the p-adic limit formula

$$\left(1 - \left(\frac{-d}{p}\right)\right) \cdot H(d) = \frac{p-1}{24} \lim_{n \to +\infty} \mathrm{Tr}(p^{2n}d). \tag{7.1}$$

If $\left(\frac{-d}{p}\right) = 1$, then this result simply implies that $\mathrm{Tr}(p^{2n}d) \to 0$ p-adically as n tends to infinity. Of course, as we saw in Theorem 4.1, it turns out that more is true. In particular, if p is any prime and $\left(\frac{-d}{p}\right) = 1$, then

$$\mathrm{Tr}(p^{2n}d) \equiv 0 \pmod{p^n}. \tag{7.2}$$

In earlier work [5], Bruinier and Ono obtained certain p-adic expansions for $H(d)$ in terms of the Borcherds exponents of certain modular functions with Heegner divisor. In his paper [14], Guerzhoy asks whether there is a connection between (7.1) and these results when $\left(\frac{-d}{p}\right) \neq 1$. In this chapter we show that this is indeed the case by establishing the following congruences. These results first appeared in [18], as joint work with Ken Ono.

Theorem 7.1. *Suppose that $-d < -4$ is a fundamental discriminant and that n is a* *positive integer. If* $p \in \{2,3\}$ *and* $\left(\frac{-d}{p}\right) = -1$, *or* $p \in \{5,7,13\}$ *and* $\left(\frac{-d}{p}\right) \neq 1$, *then*

$$\frac{24}{p-1} \cdot \left(1 - \left(\frac{-d}{p}\right)\right) \cdot H(d) \equiv \mathrm{Tr}(p^{2n}d) \pmod{p^n}.$$

In particular, under these hypotheses p^n *divides* $\frac{24}{p-1}\left(1 - \left(\frac{-d}{p}\right)\right) \cdot H(d)$ *if and only if* p^n *divides* $\mathrm{Tr}(p^{2n}d)$.

Remark. Theorem 7.1 includes $p = 2$. For simplicity, Guerzhoy chose to work with odd primes p, and this explains the omission of $p = 2$ in (7.1).

Remark. Despite the uniformity of (7.2), it turns out that the restriction on p in Theorem 7.1 is required. For example, if $p = 11$, $n = 1$ and $-d = -15$, then $\left(\frac{-15}{11}\right) = -1$, $H(-15) = 2$, and we have

$$\begin{aligned} \mathrm{Tr}(11^2 \cdot 15) &= -13374447806956269126908865521582974841084501554961922745794 \\ &\equiv 7 \not\equiv \frac{48}{10} \cdot H(-15) \pmod{11}. \end{aligned}$$

Remark. There are generalizations of Theorem 7.1 which hold for primes $p \notin \{2,3,5,7,13\}$. For example, one may employ Serre's theory [22] of p-adic modular forms to derive more precise versions of Corollary 2.4 (b) of [14].

7.1 Proving divisibility criteria

The proof of Theorem 7.1 follows by combining earlier work of Bruinier and Ono with Zagier's modularity theorem and Theorem 4.1.

For fundamental discriminants $-d < -4$, Borcherds' theory on the infinite product expansion of modular forms with Heegner divisor [3] implies that

$$q^{-H(d)} \prod_{n=1}^{\infty} (1 - q^n)^{A_1(n^2,d)}$$

is a weight zero modular function on $\mathrm{SL}_2(\mathbb{Z})$ whose divisor consists of a pole of order $H(d)$ at infinity and a simple zero at each Heegner point of discriminant $-d$. Using this factorization, Bruinier and the second author proved the following theorem.

Theorem 7.2 ([9], Corollary 3). *Let $-d < -4$ be a fundamental discriminant. If $p \in \{2,3\}$ and $\left(\frac{-d}{p}\right) = -1$, or $p \in \{5,7,13\}$ and $\left(\frac{-d}{p}\right) \neq 1$, then as p-adic numbers we have*

$$H(d) = \frac{p-1}{24}\sum_{k=0}^{\infty} p^k A_1(p^{2k}, d).$$

Remark. The case when $p = 13$ is not proven in [9]. However, thanks to the remark preceding Theorem 8 of [22] on 13-adic modular forms with weight congruent to $2 \pmod{12}$, and Theorem 2 of [9], the proof of Corollary 3 [9] still applies *mutatis mutandis*.

Recall that Zagier's modularity theorem (Theorem 1.3) can be written as follows.

Theorem 7.3. *For all positive integers $d \equiv 0, 3 \pmod 4$,*

$$\mathrm{Tr}(d) = A_1(1, d).$$

Applying Zagier duality, Theorem 4.1 can be written in the following form.

Theorem 7.4. *If p is a prime and d, D, n are positive integers such that $-d, D \equiv 0, 1 \pmod 4$, then*

$$\begin{aligned} A_1(D, p^{2n}d) =& p^n A_1(p^{2n}D, d) + \sum_{k=0}^{n-1}\left(\frac{D}{p}\right)^{n-k-1}\left(A_1\left(\frac{D}{p^2}, p^{2k}d\right) - p^{k+1}A_1\left(p^{2k}D, \frac{d}{p^2}\right)\right) \\ &+ \sum_{k=0}^{n-1}\left(\frac{D}{p}\right)^{n-k-1}\left(\left(\left(\frac{D}{p}\right) - \left(\frac{-d}{p}\right)\right)p^k A_1(p^{2k}D, d)\right). \end{aligned}$$

Proof of Theorem 7.1. Under the given hypotheses, Theorem 7.2 implies that

$$\frac{24}{p-1}\cdot H(d) \equiv \sum_{k=0}^{n-1} p^k A_1(p^{2k}, d) \pmod{p^n}. \tag{7.3}$$

By letting $D = 1$ in Theorem 7.4, for these d and p we find that

$$\left(1 - \left(\frac{-d}{p}\right)\right) \sum_{k=0}^{n-1} p^k A_1(p^{2k}, d) = A(1, p^{2n} d) - p^n A_1(p^{2n}, d). \tag{7.4}$$

Inserting this expression for the sum into (7.3), we conclude that

$$\frac{24}{p-1} \cdot \left(1 - \left(\frac{-d}{p}\right)\right) \cdot H(d) \equiv A_1(1, p^{2n} d) \pmod{p^n},$$

which by Zagier's theorem is $\mathrm{Tr}(p^{2n} d)$. $\square$

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