ANNALES
POLONICI MATHEMATICI
101.2 (2011)

# Harmonic mappings onto parallel slit domains

by Michael Dorff (Provo, UT), Maria Nowak (Lublin)
and Magdalena Wołoszkiewicz (Lublin)

**Abstract.** We consider typically real harmonic univalent functions in the unit disk $\mathbb{D}$ whose range is the complex plane slit along infinite intervals on each of the lines $x \pm ib$, $b > 0$. They are obtained via the shear construction of conformal mappings of $\mathbb{D}$ onto the plane without two or four half-lines symmetric with respect to the real axis.

**1. Introduction.** Let $S_H$ be the class of functions $f$ that are univalent sense-preserving harmonic mappings of the unit disk $\mathbb{D} = \{z : |z| < 1\}$ and satisfy $f(0) = 0$ and $f_z(0) > 0$. Next let $S_H^0$ be the subclass of $S_H$ consisting of $f$ with $f_{\bar{z}}(0) = 0$. Since harmonic mappings in $S_H^0$ are not determined by their image domains, many authors have studied subclasses of $S_H^0$ consisting of functions mapping $\mathbb{D}$ onto a specific simply connected domain $\Omega$. In particular, in [6] Hengartner and Schober considered the case of $\Omega$ being the horizontal strip $\{w : |\operatorname{Im} w| < \pi/4\}$. Later Dorff [2] considered the case of $\Omega$ being an asymmetric vertical strip, and Livingston [7] considered the case of $\Omega$ being the plane $\mathbb{C}$ slit along the interval $(-\infty, a]$, $a < 0$. Also Livingston [8], and Szapiel and Grigoryan [5] studied the case when $\Omega$ is $\mathbb{C} \setminus (-\infty, a] \cup [b, \infty)$.

Here we consider the case when a simply connected domain $\Omega$ is the plane slit along infinite intervals on each of the lines $x \pm ib$ with some $b > 0$. Let $S_H^R(\mathbb{D}, \Omega) \subset S_H^0$ be the class of harmonic typically real functions $f$ mapping the disk $\mathbb{D}$ onto $\Omega$. Since the domain $\Omega$ is convex in the horizontal direction, as in the cases mentioned above, the shear construction introduced by Clunie and Sheil-Small can be applied. In our case the so-called conformal preshear $Q$ is typically real and maps the disk onto the plane without two or four half-lines symmetric with respect to the real axis. In the next section we study the properties of the function $Q$ and, in particular, we find the preimages of horizontal lines $\operatorname{Im} Q = \alpha$. We also define a family $\mathcal{F}$ of harmonic

2010 *Mathematics Subject Classification*: Primary 30C55, 31A05.
*Key words and phrases*: harmonic mappings, shear construction.

DOI: 10.4064/ap101-2-5  

mappings such that $S_H^R(\mathbb{D},\Omega)\subset\mathcal{F}$. We discuss properties of functions from the family $\mathcal{F}$ and present several examples of harmonic functions from $\mathcal{F}$.

**2. Conformal preshear.** We start with the following

LEMMA 2.1. *For $A,B>0$ and $c\in[-2,2]$, the function $Q(z)$ defined by*

$$Q(z)=A\log\frac{1+z}{1-z}+B\frac{z}{1+cz+z^2} \tag{2.1}$$

*is a univalent map of $\mathbb{D}$ onto a domain convex in the direction of the real axis.*

*Proof.* We will show that $iQ(z)$ maps $\mathbb{D}$ onto a domain convex in the direction of the imaginary axis. By the result of Royster and Ziegler [9], it suffices to show that there are numbers $\mu\in[0,2\pi)$, $\gamma\in[0,\pi]$, such that

$$\mathrm{Re}\{e^{i\mu}(1-2\cos\gamma\, e^{-i\mu}z+e^{-2i\mu}z^2)Q'(z)\}\geq 0,\qquad z\in\mathbb{D}.$$

Choosing $\mu=0$ and $\gamma\in[0,\pi]$ so that $\cos\gamma=-c/2\in[-1,1]$ implies that the left-hand side of the last inequality is equal to

$$\begin{aligned}\mathrm{Re}&\bigg\{(1+cz+z^2)\bigg(2A\frac{1}{1-z^2}+B\frac{1-z^2}{(1+cz+z^2)^2}\bigg)\bigg\}\\ &=\bigg(\frac{2A}{|1-z^2|^2}+\frac{B}{|1+cz+z^2|^2}\bigg)(1-|z|^2)(1+|z|^2+c\,\mathrm{Re}(z)).\end{aligned}$$

So the result follows from the fact that $c\in[-2,2]$. ∎

We remark that in the case when $A=\frac{1}{2}\sin^2\alpha$, $B=\cos^2\alpha$, $\alpha\in(0,\pi/2)$, and $c=-2$, Lemma 2.1 was proved in [4] where the authors also studied classes of harmonic mappings obtained by shearing these functions.

A calculation shows that in the case of $c=2$ the image of the unit disk under $Q$ is

$$\mathbb{C}\setminus\bigg\{x\pm\frac{A\pi}{2}i : x\in\bigg[-\frac{A}{2}\log\frac{2A}{B}+\frac{2A+B}{4},\infty\bigg)\bigg\},$$

while for $c=-2$ the image is

$$\mathbb{C}\setminus\bigg\{x\pm\frac{A\pi}{2}i : x\in\bigg(-\infty,\frac{A}{2}\log\frac{2A}{B}-\frac{2A+B}{4}\bigg]\bigg\}.$$

In the case when $c\in(-2,2)$ the function $Q$ maps the unit disk onto the complex plane minus four horizontal half-lines. In particular, if $c=0$, then the resulting image is the $\mathbb{C}$ plane without the four symmetric half-lines

$$\bigg\{x\pm\frac{A\pi}{2}i : x\in\bigg(-\infty,-\frac{A}{2}\log\bigg(\frac{\sqrt{2A+B}+\sqrt{B}}{\sqrt{2A+B}-\sqrt{B}}\bigg)-\frac{\sqrt{B(2A+B)}}{2}\bigg]\bigg\}$$

and

$$\left\{x \pm \frac{A\pi}{2} i : x \in \left[\frac{A}{2} \log\left(\frac{\sqrt{2A+B}+\sqrt{B}}{\sqrt{2A+B}-\sqrt{B}}\right) + \frac{\sqrt{B(2A+B)}}{2}, \infty\right)\right\}.$$

Assume now that $Q$ is given by (2.1) with $c = -2\cos\gamma$, $\gamma \in (0, \pi)$. Then, setting $\eta = e^{i\gamma}$, we have

$$Q(z) = A \log \frac{1+z}{1-z} + B \frac{z}{(1-\eta z)(1-\overline{\eta} z)}. \tag{2.2}$$

Our aim is now to study the preimages of the horizontal lines $\operatorname{Im} Q = \alpha > 0$. Using the transformation $\zeta = \zeta(z) = \frac{1+z}{1-z}$ we can write

$$Q(z) = A \log \zeta + B \frac{\zeta^2 - 1}{4\sin^2\frac{\gamma}{2}\left(\zeta + i\cot\frac{\gamma}{2}\right)\left(\zeta - i\cot\frac{\gamma}{2}\right)}.$$

We put $\zeta = re^{i\theta}$ and consider the level curve

$$\begin{aligned} \operatorname{Im} Q &= A\theta + \frac{B}{4\sin^4\frac{\gamma}{2}} \frac{\sin 2\theta}{\left(r - \frac{\cot^2(\gamma/2)}{r}\right)^2 + 4\cot^2\frac{\gamma}{2}\cos^2\theta} \\ &= A\theta + \frac{B}{4\sin^4\frac{\gamma}{2}} \frac{\sin 2\theta}{\left(r + \frac{\cot^2(\gamma/2)}{r}\right)^2 - 4\cot^2\frac{\gamma}{2}\sin^2\theta} = \alpha, \end{aligned}$$

where

$$0 < \theta < \min\{\alpha/A, \pi/2\}.$$

So, the equations of these level curves in polar coordinates can be written in the form

$$\left(r - \frac{\cot^2\frac{\gamma}{2}}{r}\right)^2 = \frac{B\sin 2\theta}{4(\alpha - A\theta)\sin^4\frac{\gamma}{2}} - 4\cot^2\frac{\gamma}{2}\cos^2\theta,$$

or

$$\left(r + \frac{\cot^2\frac{\gamma}{2}}{r}\right)^2 = \frac{B\sin 2\theta}{4(\alpha - A\theta)\sin^4\frac{\gamma}{2}} + 4\cot^2\frac{\gamma}{2}\sin^2\theta.$$

Consequently,

$$\begin{aligned} r - \frac{\cot^2\frac{\gamma}{2}}{r} &= \pm 2\cot\frac{\gamma}{2}\cos\theta\sqrt{\frac{B\tan\theta}{2(\alpha - A\theta)\sin^2\gamma} - 1} \\ &= \pm\cot\frac{\gamma}{2}\sqrt{\frac{B\sin 2\theta}{(\alpha - A\theta)\sin^2\gamma} - 4\cos^2\theta}, \end{aligned} \tag{2.3}$$

and

$$r + \frac{\cot^2 \frac{\gamma}{2}}{r} = 2\cot\frac{\gamma}{2}\sin\theta\sqrt{\frac{B\cot\theta}{2(\alpha - A\theta)\sin^2\gamma} + 1} \tag{2.4}$$
$$= \cot\frac{\gamma}{2}\sqrt{\frac{B\sin 2\theta}{(\alpha - A\theta)\sin^2\gamma} + 4\sin^2\theta}.$$

We assume first that $\alpha > \pi A/2$ and show that preimage of $\operatorname{Im} Q = \alpha$ in the $z$-plane is a Jordan curve passing through the point $\eta$ and except for this point lying in the upper half of $\mathbb{D}$. It follows from (2.3) that $\theta \in (\theta_0, \pi/2)$, where $\theta_0$ satisfies the equation

$$\frac{B\tan\theta}{2(\alpha - A\theta)\sin^2\gamma} = 1.$$

It follows from (2.3) and (2.4) that

$$r = \frac{1}{2}\cot\frac{\gamma}{2}\Bigg(\sqrt{\frac{B\sin 2\theta}{(\alpha - A\theta)\sin^2\gamma} + 4\sin^2\theta} \pm \sqrt{\frac{B\sin 2\theta}{(\alpha - A\theta)\sin^2\gamma} - 4\cos^2\theta}\Bigg), \tag{2.5}$$

where $\theta \in (\theta_0, \pi/2)$. On the other hand,

$$\operatorname{Re} Q = A\log r + \frac{B}{4\sin^2\frac{\gamma}{2}}\,\frac{\left(r - \frac{\cot(\gamma/2)}{r}\right)\left(r + \frac{\cot(\gamma/2)}{r}\right) + \cos 2\theta\left(\cot^2\frac{\gamma}{2} - 1\right)}{\left(r - \frac{\cot^2(\gamma/2)}{r}\right)^2 + 4\cot^2\frac{\gamma}{2}\cos^2\theta}. \tag{2.6}$$

It follows from the above that the first term in (2.6) is bounded and a calculation gives that the second term is equal to

$$\frac{1}{2\sin^2\gamma}\Big(B\cos\gamma \pm \sqrt{B + 2(\alpha - A\theta)\sin^2\gamma\tan\theta}\sqrt{B - 2(\alpha - A\theta)\sin^2\gamma\cot\theta}\Big). \tag{2.7}$$

This shows that $\operatorname{Re} Q$ tends to $\pm\infty$ if $\theta$ tends to $\pi/2$, which means that the preimage of the level curve $\operatorname{Im} Q = \alpha$ in the $\zeta$-plane is a Jordan curve passing through the point $i\cot(\gamma/2)$ lying in the first quadrant except for this point and our claim is proved.

Assume now that $0 < \alpha < A\pi/2$. Then the preimage of the level curve $\operatorname{Im} Q = \alpha$ in the $\zeta$-plane in polar coordinates is also given by (2.5), where $\theta \in (\theta_0, \alpha/A)$. This implies that if $\theta$ tends to $\alpha/A$, then $r$ tends to either 0 or $\infty$. Moreover, by (2.7) the second term in the sum on the right-hand side of equation (2.6) is bounded for $\theta \in (\theta_0, \alpha/A)$. This means that the preimage of the level curve $\operatorname{Im} Q = \alpha$ in the $\zeta$-plane is a regular line going from zero to infinity which corresponds to a curve connecting 1 and $-1$ in the upper half of $\mathbb{D}$ in the $z$-plane.

Finally we note that the preimage of an interval lying on the line $\operatorname{Im} w = A\pi/2$ is a curve joining two boundary points of $\mathbb{D}$ where the derivative of $Q$ vanishes.

We have already mentioned that in the case when $c = 2, -2$, the function $Q$ maps the unit disk onto the plane slit along two parallel horizontal half-lines. In the manner used above but with less tedious calculations one can show that in these cases preimages of the horizontal lines $\operatorname{Im} Q = \alpha$ are curves connecting 1 and $-1$ for $0 < \alpha < A\pi/2$ and Jordan curves passing through $-1$ (resp. 1) for $\alpha > A\pi/2$.

**3. The class $S_H^R(\mathbb{D}, \Omega)$.** Let $\Omega$ and $S_H^R(\mathbb{D}, \Omega)$ be as in the Introduction and assume that $f \in S_H^R(\mathbb{D}, \Omega)$. Next, let $F$ and $G$ be functions analytic in $\mathbb{D}$ satisfying

$$F(0) = G(0) = 0, \qquad \operatorname{Re} f(z) = \operatorname{Re} F(z), \qquad \operatorname{Im} f(z) = \operatorname{Im} iG(z).$$

If

$$h = (F + iG)/2 \quad \text{and} \quad g = (F - iG)/2$$

then

$$f = h + \overline{g} \quad \text{and} \quad |g'(z)| < |h'(z)|.$$

Moreover, the function $h - g = iG$ is univalent, convex in the horizontal direction, and $G(\mathbb{D})$ is $\mathbb{C}$ slit along one or two infinite rays on the vertical lines $x = \pm b$. We also note that $f$ is typically real if and only if $iG = h - g$ is typically real. So the image of $\mathbb{D}$ under $iG$ is symmetric with respect to the real axis.

It follows from the above that

$$iG(z) = Q(z) = A \log \frac{1+z}{1-z} + B \frac{z}{1 + cz + z^2},$$

where $A, B > 0$, $c \in [-2, 2]$. We also note that $A = 2b/\pi$.

Consequently,

$$F(z) = h(z) + g(z) = \int_0^z \frac{h'(\zeta) + g'(\zeta)}{h'(\zeta) - g'(\zeta)} (h'(\zeta) - g'(\zeta))\, d\zeta = \int_0^z iG'(\zeta) P(\zeta)\, d\zeta,$$

where $P$ is in the class $\mathcal{P}$ of functions analytic in $\mathbb{D}$ with $P(0) = 1$ and $\operatorname{Re} P(z) > 0$ for $z \in \mathbb{D}$.

Thus

$$f(z) = \operatorname{Re}\left\{\int_0^z \left(\frac{2A}{1-\zeta^2} + B\frac{1-\zeta^2}{(1+c\zeta+\zeta^2)^2}\right)P(\zeta)\,d\zeta\right\} + i\operatorname{Im}\left\{A\log\frac{1+z}{1-z} + B\frac{z}{1+cz+z^2}\right\}.$$

Using the function

$$Q_{A,B,c}(z) = A\log\frac{1+z}{1-z} + B\frac{z}{1+cz+z^2}$$

the last formula can be written in the form

$$f(z) = \operatorname{Re}\int_0^z Q'_{A,B,c}(\zeta)P(\zeta)\,d\zeta + i\operatorname{Im}Q_{A,B,c}(z). \tag{3.1}$$

Now we define the family

$$\mathcal{F} = \left\{f : f(z) = \operatorname{Re}\int_0^z Q'_{A,B,c}(\zeta)P(\zeta)\,d\zeta + i\operatorname{Im}Q_{A,B,c}(z), \quad A, B > 0,\ c \in [-2,2],\ P \in \mathcal{P}\right\}.$$

So, we have

Theorem 3.1. *$S_H^R(\mathbb{D}, \Omega) \subset \mathcal{F}$.*

The next theorem gives one of the properties of the family $\mathcal{F}$ that can be proved using the method applied by Hengartner and Schober [6] and Grigorian and Szapiel [5] and others. We include its proof for the reader's convenience.

Theorem 3.2. *For each $f \in \mathcal{F}$, every horizontal line has a non-empty connected intersection with the image $f(\mathbb{D})$.*

*Proof.* Let $f \in \mathcal{F}$, $f = h + \overline{g} = \operatorname{Re}(h+g) + i\operatorname{Im}(h-g)$. Let $\Omega = Q(\mathbb{D})$. We consider the images of horizontal lines contained in $\Omega$ under the function $f \circ Q^{-1}$. We observe that in the case when $\alpha \neq \pm b$ the entire line $\{w = t + i\alpha : t \in \mathbb{R}\}$ is contained in $\Omega$ while $\{w = t \pm ib : t \in \mathbb{R}\} \cap Q(\mathbb{D})$ are finite or infinite intervals. Note first that

$$\operatorname{Im}[f(Q^{-1}(t+i\alpha))] = \operatorname{Im}[Q(Q^{-1}(t+i\alpha))] = \alpha,$$

so the function $f \circ Q^{-1}$ maps horizontal lines into themselves. Moreover,

$$\begin{aligned}\frac{\partial}{\partial t}[f(Q^{-1}(t+i\alpha))] &= \frac{\partial}{\partial t}[\operatorname{Re}(f(Q^{-1}(t+i\alpha)))]\\ &= \operatorname{Re}\big(Q'(Q^{-1}(t+i\alpha))P(Q^{-1}(t+i\alpha))(Q^{-1}(t+i\alpha))'\big)\\ &= \operatorname{Re}(P(Q^{-1}(t+i\alpha))) > 0.\end{aligned}$$

Thus the functions $t \mapsto \operatorname{Re}(f \circ Q^{-1}(t + i\alpha))$ are strictly increasing for each $\alpha \in \mathbb{R}$. Therefore every horizontal line has a non-empty intersection with $f(\mathbb{D})$. ■

In the next theorem we give some sufficient conditions for the containment of the entire horizontal lines $\operatorname{Im} z = \alpha$ ($\alpha \neq \pm b$) in $f(\mathbb{D})$.

THEOREM 3.3. *Assume that $Q$ is given by* (2.2) *with $\eta = e^{i\gamma}$ and $f$ is defined by* (3.1)*. Let $\gamma \in [0, \pi]$. If the function $P$ in* (3.1) *is analytic at $\eta$ and $\operatorname{Re} P(\eta) > 0$, then the half-plane $\{w : \operatorname{Im} w > b\}$ is contained in $f(\mathbb{D})$. If the function $P$ is analytic at $\bar{\eta}$ and $\operatorname{Re} P(\bar{\eta}) > 0$, then the half-plane $\{w : \operatorname{Im} w < -b\}$ is contained in $f(\mathbb{D})$. Finally, if the function $P$ is analytic at $1$ and $-1$, $\operatorname{Re} P(1) > 0$ and $\operatorname{Re} P(-1) > 0$, then the horizontal strip $\{w : |\operatorname{Im} w| < b\}$ is contained in $f(\mathbb{D})$.*

*Proof.* Assume $P$ is analytic at $\eta$ and $\operatorname{Re} P(\eta) > 0$. Consider the function

$$F(z) = \int_0^z Q'(\zeta)P(\zeta)\, d\zeta, \tag{3.2}$$

where $Q$ is given by (2.2). Then in a neighborhood of $\eta$, when $\eta \neq \pm 1$,

$$F'(z) = P(\eta)Q'(z) + \left(P'(\eta)(z-\eta) + \frac{P''(\eta)}{2}(z-\eta)^2 + \cdots\right) \times \left(\frac{-B\eta}{(\eta-\bar{\eta})(z-\eta)^2} + \frac{a_{-1}}{z-\eta} + a_0 + \cdots\right),$$

and when $\eta^2 = 1$,

$$F'(z) = P(\eta)Q'(z) + \left(P'(\eta)(z-\eta) + \frac{P''(\eta)}{2}(z-\eta)^2 + \cdots\right) \times \left(\frac{-2B\eta}{(z-\eta)^3} - \frac{B}{(z-\eta)^2} + \frac{a_{-1}}{z-\eta} + a_0 + \cdots\right).$$

Thus the function $w_\eta$ defined by

$$w_\eta(z) = \begin{cases} F(z) - P(\eta)Q(z) + \dfrac{B\eta P'(\eta)}{\eta - \bar{\eta}} \log(1 - \bar{\eta}z) & \text{if } \eta^2 \neq 1, \\ F(z) - P(\eta)Q(z) & \\ \qquad - B(P'(\eta) + \eta P''(\eta)) \log \dfrac{1}{1-\eta z} + \dfrac{2BP'(\eta)}{1-\eta z} & \text{if } \eta^2 = 1, \end{cases}$$

is analytic at $\eta$. Consequently, in the case $\eta^2 \neq 1$,

$$\begin{aligned} F(z) &= F(z) - w_\eta(z) + w_\eta(z) \\ &= Q(z)\left(P(\eta) - \frac{B\eta P'(\eta)(1-\eta z)(1-\bar{\eta}z)\log(1-\bar{\eta}z)}{(\eta-\bar{\eta})(A(1-\bar{\eta}z)(1-\eta z)\log\frac{1+z}{1-z} + Bz)}\right) + w_\eta(z), \end{aligned}$$

and in the case $\eta^2 = 1$,

$$F(z) = Q(z)\left(P(\eta) + \frac{B\big((P'(\eta) + \eta P''(\eta)) \log \frac{1}{1-\eta z} - \frac{2P'(\eta)}{1-\eta z}\big)(1-\eta z)^2}{A(1-\eta z)^2 \log \frac{1+z}{1-z} + Bz}\right) + w_\eta(z).$$

Therefore,

$$F(z) = Q(z)(P(\eta) + o(1)) + w_\eta(z) \quad \text{as } z \to \eta.$$

It follows from the work in Section 2 that the preimages $\Gamma_\alpha$ of the lines

$$\operatorname{Im} f(z) = \operatorname{Im} Q(z) = \alpha > b \quad \text{or} \quad \operatorname{Im} f(z) = \operatorname{Im} Q(z) = \alpha < -b$$

are curves in $\mathbb{D}$ that approach $\eta$ or $\bar{\eta}$, respectively. Since

$$\operatorname{Re} f(z) = \operatorname{Re} F(z),$$

we see that $\operatorname{Re} f(z)$ converges to $\pm\infty$ as $z$ approaches $\eta$ or $\bar{\eta}$ along $\Gamma_\alpha$.

Assume now that $\eta = e^{i\gamma}$ with $\gamma \in (0, \pi)$. If the function $P$ is analytic at 1 and $-1$, $\operatorname{Re} P(1) > 0$, and $\operatorname{Re} P(-1) > 0$, then $w_1(z) = F(z) - P(1)Q(z)$ is analytic at 1 and $w_{-1}(z) = F(z) - P(-1)Q(z)$ is analytic at $-1$. This means that $\operatorname{Re} f(z) = \operatorname{Re} F(z)$ behaves as $\operatorname{Re} Q(z)$ near 1 and $-1$. Moreover, we know from Section 2 that preimages of the lines

$$\operatorname{Im} f(z) = \operatorname{Im} Q(z) = \alpha, \qquad \text{where } |\alpha| < b,$$

are curves in $\mathbb{D}$ connecting 1 and $-1$. So, our claim follows. The same conclusion can be drawn for the cases when $\eta = 1$ and $\eta = -1$. ■

Corollary 3.4. *If $f \in \mathcal{F}$ has dilatation $\omega(z) = g'(z)/h'(z)$ such that $|\omega(z)| \le C < 1$ for $z \in \mathbb{D}$, then the complement of $f(\mathbb{D})$ consists of infinite intervals lying on two parallel lines $z = \pm ib$.*

For fixed $A, B > 0$, $c \in [-2, 2]$ let $\mathcal{F}(A, B, c)$ denote the subset of $\mathcal{F}$ with $Q = Q_{A,B,c}$. As we noted before, the class $\mathcal{F}(A, B, c)$ contains the harmonic univalent maps of the disk $\mathbb{D}$ onto the plane slit along the horizontal lines $z = \pm ib$, where $b = \pi A/2$. Now for fixed $b > 0$ (or equivalently $A > 0$) let

$$\mathcal{F}(b) = \bigcup_{B>0,\, -2\le c\le 2} \mathcal{F}(A, B, c)$$

and let $S_H^R(b)$ denote the class of typically real univalent harmonic mappings of the disk $\mathbb{D}$ onto the plane slit along the horizontal lines $z = \pm ib$. We have the following.

Corollary 3.5. *For $b > 0$,*

$$\overline{S_H^R(b)} = \mathcal{F}(b).$$

*Proof.* Let $f \in \mathcal{F}(b)$ be given by (3.1) with some $P \in \mathcal{P}$. For an integer $n > 2$ define $P_n(z) = P((1-1/n)z)$ and set

$$f_n(z) = \operatorname{Re}\int_0^z Q'(\zeta)P_n(\zeta)\,d\zeta + i\operatorname{Im} Q(z).$$

By Theorem 3.3, $f_n \in S_H^R(b)$ and the sequence $\{f_n\}$ converges locally uniformly on $\mathbb{D}$ to $f$. ■

The next theorem describes situations when functions $f$ from the family $\mathcal{F}$ have the property that the intersections of horizontal lines with $f(\mathbb{D})$ are finite intervals.

THEOREM 3.6. *Assume that $Q$ is given by* (2.2) *with $\eta = e^{i\gamma}$, $\gamma \in (0,\pi)$, and $f$ is defined by* (3.1). *If the function $P$ in* (3.1) *is analytic at $\eta$ ($\bar\eta$) and $P(\eta) = 0$ ($P(\bar\eta) = 0$), then the intersection of every horizontal line* $\operatorname{Im} w = \alpha$, *$\alpha > b$ ($\alpha < -b$), with $f(\mathbb{D})$ is a finite interval. Moreover, if the function $P$ is analytic at* 1 *and* $-1$, *and* $P(1) = P(-1) = 0$, *then the intersection of a horizontal line* $\operatorname{Im} w = \alpha$ *($|\alpha| < b$) with $f(\mathbb{D})$ is a finite interval.*

*Proof.* Assume that $P$ is analytic at $\eta$, $P(\eta) = 0$ and $F$ is given by (3.2). Then in a neighborhood of $\eta$,

$$F'(z) = -\frac{B\eta P'(\eta)}{(\eta-\bar\eta)(z-\eta)} + w_\eta(z),$$

where $w_\eta$ is analytic at $\eta$. Consequently,

$$F(z) = \frac{B\eta P'(\eta)}{\eta-\bar\eta}\log\frac{1}{1-\bar\eta z} + W_\eta(z),$$

with $W_\eta$ analytic at $\eta$. It has been noted in [5, pp. 66–67] that $\eta P'(\eta) < 0$. Hence in a neighborhood of $\eta$,

$$\operatorname{Re} f(z) = \operatorname{Re} F(z) = \operatorname{Im}\left(\frac{B\eta P'(\eta)}{2\sin\gamma}\log\frac{1}{1-\bar\eta z}\right) + \operatorname{Re} W_\eta(z).$$

Now our claim follows from the properties of the set $\{z \in \mathbb{D} : \operatorname{Im} f(z) = \alpha\}$ for $\alpha > b$. The other statement can be proved by observing that if $P$ is analytic at 1 and $-1$, and $P(1) = P(-1) = 0$, then $F$ is analytic at 1 and $-1$. ■

We note that the assertion of Theorem 3.6 does not hold in the case $\eta = \pm 1$. In particular, if $\eta = 1$, $P$ is analytic at 1 and $P(1) = 0$, then the intersection of every horizontal line $\operatorname{Im} w = \alpha$ ($\alpha > b$) with $f(\mathbb{D})$ is either this line or a half-line $\{w : w = x + i\alpha,\ x > x_\alpha\}$ with some real $x_\alpha$. Indeed, if

$$Q(z) = A\log\frac{1+z}{1-z} + B\frac{z}{(1-z)^2}$$

and $F$ is defined by (3.2), then

$$F(z)=\frac{2BP'(1)}{(z-1)}+B(P'(1)+P''(1))\log\frac{1}{z-1}+w(z),$$

where $w$ is analytic at 1. Hence

$$\operatorname{Re}F(z)=2BP'(1)\operatorname{Re}\frac{1}{z-1}+B(P'(1)+\operatorname{Re}P''(1))\log\frac{1}{|z-1|}+O(1)$$

as $\mathbb{D}\ni z\to 1$. Using the transformation $\zeta=\zeta(z)=\frac{1+z}{1-z}$ we can write $\operatorname{Re}F(\zeta)=-BP'(1)\operatorname{Re}\zeta+B(P'(1)+\operatorname{Re}P''(1))\log|\zeta+1|+O(1)$ as $\zeta\to\infty$. A calculation shows that the preimage of the level curve $\operatorname{Im}f=\operatorname{Im}Q=\alpha>b$ in the $\zeta$-plane can be written in the form

$$r=2\sqrt{\frac{\alpha-A\theta}{B\sin 2\theta}}, \tag{3.3}$$

where $\zeta=re^{i\theta}$, $\theta\in(0,\pi/2)$. It has been proved in [5] that $P'(1)+\operatorname{Re}P''(1)\leq 0$. We now show that if we assume additionally that $P'(1)+\operatorname{Re}P''(1)=0$, then $f(\mathbb{D})$ contains the half-lines described above. Indeed, on the curve given by (3.3) we have

$$\operatorname{Re}F(\zeta)=-BP'(1)\cdot 2\sqrt{\frac{\alpha-A\theta}{B\sin 2\theta}}\cos\theta+O(1)$$

and our claim follows from the fact that

$$\lim_{\theta\to 0^+}2\sqrt{\frac{\alpha-A\theta}{B\sin 2\theta}}\cos\theta=+\infty\quad\text{and}\quad\lim_{\theta\to\pi/2^-}2\sqrt{\frac{\alpha-A\theta}{B\sin 2\theta}}\cos\theta=0.$$

Similar analysis can be used to show that if $P'(1)+\operatorname{Re}P''(1)<0$, then $f(\mathbb{D})$ contains the whole horizontal lines $\operatorname{Im}w=\alpha>b$.

**4. Examples.** In this section we give examples of harmonic functions from the family $\mathcal{F}$. Our first example is a harmonic map of the unit disk onto the complex plane slit along four horizontal half-lines that are symmetric with respect to the real axis.

EXAMPLE 4.1. Let $Q_1=Q_{1/4,1/2,0}$ and take $P(z)=\frac{1+z^4}{1-z^4}$. Then we obtain

$$\begin{aligned}f_1(z)=\;&\operatorname{Re}F_1(z)+i\operatorname{Im}Q_1(z)\\=\;&\operatorname{Re}\bigg(-\frac{5i}{16}\log\Big(\frac{1+iz}{1-iz}\Big)+\frac{1}{4}\frac{z}{1-z^2}-\frac{1}{8}\frac{z}{1+z^2}+\frac{1}{4}\frac{z}{(1+z^2)^2}\bigg)\\&+i\operatorname{Im}\bigg(\frac{1}{4}\log\Big(\frac{1+z}{1-z}\Big)+\frac{1}{2}\frac{z}{1+z^2}\bigg).\end{aligned}$$

We will show that the function $f_1$ maps the unit disk onto the plane minus four parallel slits given by $\{x\pm i\pi/8:|x|\geq 5\pi/32\}$. We will use a similar argument to that applied by Clunie and Sheil-Small [1] for the so-called harmonic Koebe function. Using the transformation $\zeta = \zeta(z) = \frac{1+z}{1-z} = \xi + i\eta$, $\xi > 0$, we get

$$f_1(z) = \operatorname{Re}\left(-\frac{5i}{16}\log\left(\frac{\zeta - i}{1 - i\zeta}\right) + \frac{1}{16}\left(\zeta - \frac{1}{\zeta}\right) + \frac{1}{8}\frac{(\zeta^2-1)\zeta}{(\zeta^2+1)^2}\right) + i\operatorname{Im}\left(\frac{1}{4}\log\zeta + \frac{1}{4}\frac{\zeta^2-1}{\zeta^2+1}\right).$$

We observe that the transformation $z \mapsto \zeta(z)$ maps the part of the disk in the first quadrant onto the exterior of the unit disk contained in the first quadrant, and we note that the interval $[0, i)$ is mapped onto the quarter of the unit circle. If we put $\zeta = re^{i\theta}$, $r \geq 1$, $\theta \in [0, \pi/2)$, then we have

$$\begin{aligned}\operatorname{Re} f_1(z) &= \frac{1}{4}\left(\frac{5}{4}\arctan\frac{r - 1/r}{2\cos\theta} + \frac{1}{4}\left(r - \frac{1}{r}\right)\cos\theta\right. \\ &\quad \left. + \frac{1}{2}\left(r - \frac{1}{r}\right)\cos\theta\frac{(r-1/r)^2 + 4(\sin^2\theta + 1)}{((r-1/r)^2 + 4\cos^2\theta)^2}\right), \\ \operatorname{Im} f_1(z) &= \frac{1}{4}\left(\theta + \frac{2\sin 2\theta}{(r-1/r)^2 + 4\cos^2\theta}\right).\end{aligned}$$

Now we consider the level curves

$$\theta + \frac{2\sin 2\theta}{(r-1/r)^2 + 4\cos^2\theta} = c, \qquad c > 0. \tag{4.1}$$

Since $r > 1$ and $\theta \in (0, \pi/2)$, we get

$$r - \frac{1}{r} = 2\cos\theta\sqrt{\frac{\tan\theta}{c-\theta} - 1}. \tag{4.2}$$

Let $\theta_c \in (0, \pi/2)$ be the number satisfying the equation $\tan\theta_c = c - \theta_c$. If $0 < c < \pi/2$, we assume that $\theta_c < \theta < c$, while if $c \geq \pi/2$, we assume that $\theta_c < \theta < \pi/2$. Fix $c > 0$. Then the image of the level curve given in (4.1) under $f_1$ is

$$\begin{aligned}f_1(z) &= \frac{1}{8}\left(\frac{5}{2}\arctan\left(\frac{\tan\theta}{c-\theta} - 1\right)^{1/2} + \cos^2\theta\left(\frac{\tan\theta}{c-\theta} - 1\right)^{1/2}\right. \\ &\quad + \frac{1}{2}\left(\frac{c-\theta}{\tan\theta}\right)^2\left(\frac{\tan\theta}{c-\theta} - 1\right)^{3/2} \\ &\quad \left. + \frac{1}{2}(c-\theta)^2\left(1 + \frac{1}{\sin^2\theta}\right)\left(\frac{\tan\theta}{c-\theta} - 1\right)^{1/2}\right) + i\frac{c}{4} \\ &= u(c, \theta) + i\frac{c}{4}.\end{aligned}$$

If $0 < c < \pi/2$, then $\theta \in (\theta_c, c)$ and we find that

$$\lim_{\theta\to\theta_c^+} u(c,\theta) = 0 \quad \text{and} \quad \lim_{\theta\to c^-} u(c,\theta) = \infty.$$

Similarly, if $c > \pi/2$, then $\theta \in (\theta_c, \pi/2)$ and we have

$$\lim_{\theta\to\theta_c^+} u(c,\theta) = 0 \quad \text{and} \quad \lim_{\theta\to\pi/2^-} u(c,\theta) = \infty.$$

Finally, if $c = \pi/2$, then $\theta \in (\theta_c, \pi/2)$ and we have

$$\lim_{\theta\to\theta_c^+} u(c,\theta) = 0 \quad \text{and} \quad \lim_{\theta\to\pi/2^-} u(c,\theta) = \frac{5\pi}{32}.$$

This means that the image under $f_1$ of the part of the disk in the first quadrant is the first quadrant minus the half-line $\{x + i\pi/8 : x \geq 5\pi/32\}$. Our claim follows from the symmetry.

In the next example we present a map onto the plane slit along two horizontal half-lines symmetric with respect to the real axis.

Example 4.2. Let $f_2$ be the harmonic shear of $Q_2 = Q_{1/8,6/8,-2}$ with $P(z) = (1+z^2)/(1-z^2)$. One can show that

$$\begin{aligned} f_2(z) &= \operatorname{Re} F_2(z) + i \operatorname{Im} Q_2(z) \\ &= \operatorname{Re}\left(\frac{1}{2}\,\frac{z(2-z+z^3)}{(1-z)^3(1+z)}\right) + i\operatorname{Im}\left(\frac{1}{8}\log\left(\frac{1+z}{1-z}\right) + \frac{6}{8}\,\frac{z}{(1-z)^2}\right). \end{aligned}$$

It was shown in [3] that $f_2$ maps the disk onto the plane minus two half-lines given by $x \pm i\pi/16$, $x \leq -1/4$.

The following two examples illustrate Theorem 3.6.

Example 4.3. Taking $Q_3 = Q_{1/4,1/2,0}$ and $P(z) = (1-z^2)/(1+z^2)$ we obtain

$$\begin{aligned} f_3(z) = {} & \operatorname{Re}\left(-\frac{3i}{8}\log\left(\frac{1+iz}{1-iz}\right) - \frac{1}{4}\,\frac{z}{1+z^2} + \frac{1}{2}\,\frac{z}{(1+z^2)^2}\right) \\ & + i\operatorname{Im}\left(\frac{1}{4}\log\left(\frac{1+z}{1-z}\right) + \frac{1}{2}\,\frac{z}{1+z^2}\right). \end{aligned}$$

Example 4.4. Let $f_4$ be the shear of $Q_4 = Q_{1/4,1/2,0}$ with $P(z) = (1-z^4)/(1+z^4)$. Then

$$f_4(z) = \operatorname{Re}\left(-\frac{i}{2}\log\left(\frac{1+iz}{1-iz}\right)\right) + i\operatorname{Im}\left(\frac{1}{4}\log\left(\frac{1+z}{1-z}\right) + \frac{1}{2}\,\frac{z}{1+z^2}\right).$$

Images of concentric circles inside $\mathbb{D}$ under $f_3$ and $f_4$ are shown in the figures below.

Our final example is a harmonic map onto the right-half plane. This map is connected with the note after Theorem 3.6.

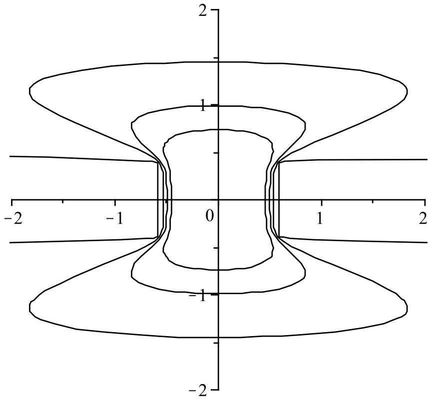


Fig. 1. Images of concentric circles inside $\mathbb{D}$ under $f_3$.

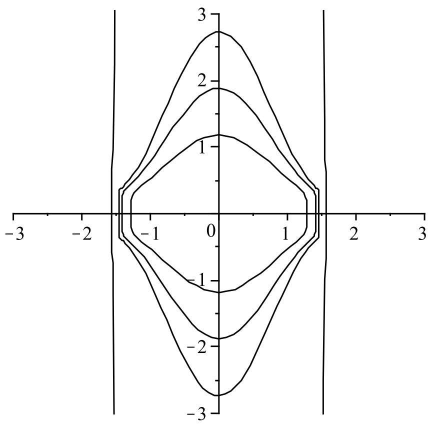


Fig. 2. Images of concentric circles inside $\mathbb{D}$ under $f_4$.

EXAMPLE 4.5. Let $Q_5 = Q_{1/4,1/2,-2}$ and take $P(z) = (1-z^2)/(1+z^2)$. Then

$$f_5(z) = \operatorname{Re}\left(\frac{z}{1-z}\right) + i\operatorname{Im}\left(\frac{1}{4}\log\frac{1+z}{1-z} + \frac{1}{2}\frac{z}{(1-z)^2}\right)$$

is the harmonic map of the disk onto the half-plane $\operatorname{Re} w > -1/2$.

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Michael Dorff
Department of Mathematics
Brigham Young University
Provo, UT 84602, U.S.A.
E-mail: mdorff@math.byu.edu

Maria Nowak, Magdalena Wołoszkiewicz
Department of Mathematics
Maria Curie-Skłodowska University
20-031 Lublin, Poland
E-mail: mt.nowak@poczta.umcs.lublin.pl
woloszkiewicz@umcs.pl

*Received 14.7.2010*
*and in final form 2.2.2011* (2252)