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Doubly close-to-convex functions

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Abstract

We introduce the class $L(\beta,\gamma)$ of holomorphic, locally univalent functions in the unit disk $\mathbb{D}=\{z\colon |z|<1\}$, which we call the class of doubly close-to-convex functions. This notion unifies the earlier known extensions. The class $L(\beta,\gamma)$ appears to be linear invariant. First of all we determine the region of variability $\{w\colon w=\log f'(r),\ f\in L(\beta,\gamma)\}$ for fixed z, $|z|=r<1$, which give us the exact rotation theorem. The rotation theorem and linear invariance allows us to find the sharp value for the radius of close-to-convexity and bound for the radius of univalence. Moreover, it was helpful as well in finding the sharp region for $\alpha\in\mathbb{R}$, for which the integral $\int_0^z (f'(t))^\alpha\,dt$, $f\in L(\beta,\gamma)$, is univalent. Because $L(\beta,\gamma)$ reduces to β-close-to-convex functions ($\gamma=0$) and to convex functions ($\beta=0$ and $\gamma=0$), the obtained results generalize several corresponding ones for these classes. We improve as well the value of the radius of univalence for the class considered by Hengartner and Schober (Proc. Amer. Math. Soc. 28 (1971) 519–524) from 0.345 to 0.577.

1. Introduction

We consider functions f that are holomorphic in $\mathbb{D}=\{z\colon |z|<1\}$ with the normalization $f(0)=0$, $f'(0)=1$ and are locally univalent in $\mathbb{D}$ (i.e., $f'(z)\neq 0$ in $\mathbb{D}$). In particular, let S denote the class of all holomorphic and univalent functions with this normalization and let $S^c\subset S$ be the subclass consisting of convex functions. A function f is said to be

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close-to-convex of order $\beta \geqslant 0$ in $\mathbb{D}$ if there exists $g \in S^c$ and $\phi \in \mathbb{R}$ such that

$$\left|\arg e^{i\phi}\frac{f'(z)}{g'(z)}\right| \leqslant \beta\frac{\pi}{2}, \quad z \in \mathbb{D}. \tag{1}$$

The class of close-to-convex functions of order β will be denoted by L_β [2,5,10]. We observe that $L_0 \equiv S^c$ and $L_1 \equiv L$, where L denotes the class of close-to-convex univalent functions in $\mathbb{D}$ [3]. If $\beta \in [0,1]$, then L_β consists of functions that are univalent only in $\mathbb{D}$ and is a linear invariant family of order $(\beta+1)$ (for all $\beta \geqslant 0$) in the sense of Pommerenke [10]. Hengartner and Schober [4] have studied the generalization of the class L by letting $\beta = 1$ and $g(z)$ to be a function which is convex in the direction of the imaginary axis in (1). Another generalization was considered in [1] and [12], where g was taken from the class of bounded boundary rotation. Here we extend these ideas by studying the more general class of doubly close-to-convex functions.

2. Doubly close-to-convex functions

Definition 1. Let $\beta \geqslant 0$ and $\gamma \geqslant 0$ be fixed. We say that a holomorphic, locally univalent function f in $\mathbb{D}$ with the normalization $f(0) = 0$, $f'(0) = 1$ belongs to the class $L(\beta,\gamma)$ if there exist $g \in L_\gamma$ and $\phi \in \mathbb{R}$ such that (1) holds. We call $L(\beta,\gamma)$ the class of doubly close-to-convex functions of order (β,γ). Of course, we have that $L(0,0) \equiv S^c$, $L(\beta,0) \equiv L_\beta$, $L(0,\gamma) \equiv L_\gamma$.

The following lemmas follow almost directly from the definition.

Lemma 1. *A function $f \in L(\beta,\gamma)$ if and only if there exists a function $h \in S^c$ and two holomorphic functions $p(z) = 1 + p_1 z + \cdots$, $q(z) = 1 + q_1 z + \cdots$ in $\mathbb{D}$ such that* $\operatorname{Re}[e^{i\phi}p(z)] > 0$ *and* $\operatorname{Re}[e^{i\psi}q(z)] > 0$ *in $\mathbb{D}$ for some $\phi,\psi \in \mathbb{R}$, and*

$$f'(z) = h'(z)p^\gamma(z)q^\beta(z). \tag{2}$$

Proof. By (1) we have $f'(z) = g'(z)q^\beta(z)$, where $g \in L_\gamma$ and $\operatorname{Re}[e^{i\psi}q(z)] > 0$, $z \in \mathbb{D}$ for some $\psi \in \mathbb{R}$ $(q(z) = 1 + q_1 z + q_2 z^2 + \cdots)$. On the other hand, $g \in L_\gamma$ if and only if $g'(z) = h'(z)p^\gamma(z)$, where $h \in S^c$ and $\operatorname{Re}[e^{i\phi}p(z)] > 0$, $z \in \mathbb{D}$ for some $\phi \in \mathbb{R}$ $(p(z) = 1 + p_1 z + \cdots)$. Therefore, we have (2). □

Remark. Formula (2) can be written in the form

$$f'(z) = h'(z)\left(\frac{1 + e^{-i\phi}\omega_1(z)}{1 - \omega_1(z)}\right)^\gamma \left(\frac{1 + e^{-i\psi}\omega_2(z)}{1 - \omega_2(z)}\right)^\beta, \tag{2'}$$

where ω_1 and ω_2 are holomorphic in $\mathbb{D}$ and satisfy the conditions of the Schwarz lemma.

Lemma 2. *The family $L(\beta,\gamma)$ is a linear invariant family of order* $(\beta+\gamma+1)$.

Proof. The proof of linear invariance is exactly the same as for L_β given in [5] or [2]. The order follows from the fact that from (1) $f(z) = z + a_2 z^2 + \cdots$, $g(z) = z + b_2 z^2 + \cdots$, and $q(z)+1+q_1z+\cdots$, we have $a_2=b_2+\frac{1}{2}\beta q_1$. Since $|b_2|\leqslant 1+\gamma$ and $|q_1|\leqslant 2$, we have $|a_2|\leqslant \beta+\gamma+1$. $\square$

The next theorem generalizes the classical result for close-to-convex functions from [6] and corresponding result from [12].

Theorem 1. *The region of variability* $G(z)=\{w\colon\ w=\log f'(z),\ f\in L(\beta,\gamma)\}$ *for fixed* $z=re^{i\phi}\in\mathbb{D}$, $0<r<1$, *is a closed and convex set whose boundary has the equation*

$$w=w(t)=\log\frac{(1-re^{i\theta_2})^{\beta+\gamma}}{(1-re^{i\theta_1})^{\beta+\gamma+2}},\quad t\in[0,2\pi],\tag{3}$$

where

$$\theta_1=\theta_1(t)=t-\arcsin(r\sin t),\qquad \theta_2=\theta_2(t)=\pi+t+\arcsin(r\sin t).\tag{4}$$

Proof. First observe that $G(z)=G(r)$, $r=|z|<1$, because the class $L(\beta,\gamma)$ is rotationally invariant. The set $G(r)$ is closed because the class $L(\beta,\gamma)$ is compact. The convexity of $G(r)$ is the consequence of the property that if $f_1, f_2\in L(\beta,\gamma)$, then for all $\lambda\in[0,1]$

$$f_\lambda(z)=\int_0^z\big[f_1(t)\big]^\lambda\big[f_2(t)\big]^{1-\lambda}\,dt\in L(\beta,\gamma).$$

Therefore, it will be enough to find the equation of the boundary of $G(r)$. By (2), it suffices to consider

$$f'(r)=h'(r)p^\gamma(r)q^\beta(r).\tag{5}$$

It is well known that the functions $h\in S^c$ corresponding to the boundary points of $\{w\colon\ w=h'(r),\ h\in S^c\}$ have the form

$$h(z)=\frac{z}{1-ze^{i\tau}},\quad \theta\in[0,2\pi],$$

and that the functions p corresponding to the boundary points of $\{w\colon\ w=p(r),\ \operatorname{Re}p(z)>0,\ z\in\mathbb{D},\ p(0)=e^{i\delta},\ |\delta|<\pi/2\}$ have the form

$$p(z)=\frac{e^{i\delta}-ze^{i(s-\delta)}}{1-ze^{is}},\quad s\in[0,2\pi].$$

The same is true for $q(z)$. These facts along with (2′) imply that the function f_0 corresponding to the boundary points of $G(r)$ has by the form

$$f_0'(r)=\frac{1}{(1-\epsilon_5r)^2}\left(\frac{1-\epsilon_1r}{1-\epsilon_2r}\right)^\gamma\left(\frac{1-\epsilon_3r}{1-\epsilon_4r}\right)^\beta,\tag{6}$$

where $\epsilon_j=e^{i\theta_j}$, $\theta_j\in[0,2\pi]$, $j=1,2,3,4,5$.

The convexity of $G(r)$ implies that finding the boundary of $G(r)$ is equivalent to determining the maximum of the function

$$\begin{aligned}\operatorname{Re}\big[e^{-it}\log f'(r)\big]=\operatorname{Re}\big\{e^{-it}\big[&-2\log(1-\epsilon_5r)+\beta\log(1-\epsilon_3r)\\&+\gamma\log(1-\epsilon_1r)-\beta\log(1-\epsilon_4r)-\gamma\log(1-\epsilon_2r)\big]\big\}\end{aligned}\tag{7}$$

with respect to $\theta_j \in [0, 2\pi]$ for fixed $t \in [0, 2\pi]$, where t denotes the angle between the imaginary axis and supporting line to $G(r)$. Moreover, we observe from (7) that $G(r)$ is symmetric with respect to the real axis, because the image of the circle $\xi = 1 - re^{i\phi}$, $\phi \in [0, 2\pi]$, under the mapping $w = \log \xi$ is a convex curve symmetric about the real axis. Therefore one can restrict considerations to $t \in [0, \pi]$.

One can verify directly that the function

$$u(\theta) = \operatorname{Re}\left\{e^{-it} \log(1 - re^{i\theta})\right\}$$

attains its maximum for $\theta_2 = \theta_2(t)$ and minimum for $\theta = \theta_1(t)$ as given in (4). □

Corollary 1. *If $f \in L(\beta, \gamma)$, then for $|z| = r < 1$ we have the following sharp bounds*:

$$\left|\arg f'(z)\right| \leqslant 2(\beta + \gamma + 1) \arcsin r, \tag{8}$$

$$\frac{(1-r)^{\beta+\gamma+2}}{(1+r)^{\beta+\gamma}} \leqslant \left|f'(z)\right| \leqslant \frac{(1+r)^{\beta+\gamma+2}}{(1-r)^{\beta+\gamma}}. \tag{9}$$

The extremal function has the form (6) *with θ_1 and θ_2 given by* (4) *with an appropriate t.*

Proof. Using the symmetry of $G(r)$ we see that the max(arg $f'(r)$) is attained for $t = \pi/2$ while the bounds for $|f'(z)|$ is attained for $t = \pi$ and $t = 0$, which implies (8) and (9). □

Theorem 2. *The radius of convexity of the class $L(\beta, \gamma)$ is equal to*

$$r_c(\beta, \gamma) = (\beta + \gamma + 1) - \sqrt{(\beta + \gamma + 1)^2 - 1}. \tag{10}$$

In particular, $r_c(1, 1) = 3 - \sqrt{8}$, $r_c(1, 0) = 2 - \sqrt{3}$ with these results being sharp.

The formula (10) follows from the Pommerenke result for linear invariant families [9, p. 133] and Lemma 2. The rotation theorem (8) and the linear invariance of the family $L(\beta, \gamma)$ determine the possibility of finding the radii of univalence and close-to-convexity for $L(\beta, \gamma)$.

Theorem 3. *The radius of univalence $r_u(\beta, \gamma)$ of the class $L(\beta, \gamma)$ satisfies the inequality $r_u(\beta, \gamma) \geqslant r_{\beta,\gamma}$, where*

$$r_{\beta,\gamma} = \tan \frac{\pi}{2(\beta + \gamma + 1)} \quad \text{if } \beta + \gamma > 1. \tag{11}$$

If $\beta + \gamma \leqslant 1$, then $r_u(\beta, \gamma) = 1$.

Corollary 2. *We have $r_u(1, 1) \geqslant \sqrt{3}/3 \cong 0.577$, which improves the corresponding result for the class considered by Hengartner and Schober in* [4], *because their class of functions is a subclass of $L(\beta, \gamma)$.* (*The constant for r_u in* [4] *was approximately* 0.345.)

Proof. If $\mathcal{M}$ is a linear invariant family, then Pommerenke [9] proved that $r_u(\mathcal{M}) \geqslant \hat{r} = r_0/(1+\sqrt{1-r_0^2})$, where $r_0 \in (0,1]$ is the radius of the disk $|z| < r_0$ in which $f(z)/z \neq 0$, $f \in \mathcal{M}$, and r_0 is determined from the equation

$$\max_{\substack{f\in\mathcal{M}\\ |z|=r<1}} \left|\arg f'(z)\right| = 2\pi.$$

From the bound in (8) we find that $r_0 = 1$ and $\hat{r} = 1$ if $\beta+\gamma \leqslant 1$ and $r_0 = \sin\frac{\pi}{\beta+\gamma+1}$ if $\beta+\gamma > 1$. By the above formula for $\hat{r} := r_{\beta+\gamma}$, Eq. (11) and Corollary 2 follow directly. □

The result of Theorem 3 can be sharpened by the exact value of the radius of close-to-convexity which is the consequence of (8) and the following less known sharp result of Campbell and Ziegler [1, p. 19] (in our formulation):

Lemma A. *If $\mathcal{M}$ is a linear invariant family for which*

$$\max_{\substack{f\in\mathcal{M}\\ |z|=r<1}} \left|\arg f'(z)\right| = 2\tau \arcsin r,$$

then the radius of close-to-convexity of $\mathcal{M}$ is 1 if $1 \leqslant \tau \leqslant 2$ and is the unique solution of the equation

$$2\operatorname{arccot} w - 2\tau \operatorname{arccot}(\tau w) = -\pi, \tag{12}$$

where

$$w = \frac{1-r^2}{\sqrt{4\tau^2 r^2 - (1+r^2)^2}} \quad \textit{if } \tau > 2.$$

Therefore we have the following sharp result.

Theorem 4. *Let $f \in L(\beta,\gamma)$. If $\beta+\gamma \leqslant 1$, then f is close-to-convex univalent in $\mathbb{D}$. If $\beta+\gamma > 1$, then the radius of close-to-convexity $r_{cc}(\beta,\gamma)$ of $L(\beta,\gamma)$ is given by* (12) *with $\tau = (\beta+\gamma+1)$.*

Corollary 3. *We have*

$$r_{cc}(1,1) = \left\{12\sqrt{3} - 19 - 2\sqrt{198 - 114\sqrt{3}}\right\}^{1/2} \cong 0.553. \tag{13}$$

Proof. When $\beta = \gamma = 1$, then $\tau = 3$ and (12) can be reduced by the formula for $\cot 3\alpha$ and after some calculations to the equation

$$t^2 - 2(2\sqrt{13} - 19)t + 1 = 0, \quad t = r^2,$$

which yields (13). This value improves the result for r_u given in [4].

Formula (13) shows that $r_u(\beta,\gamma) > r_{cc}(\beta,\gamma)$ for the class $L(\beta,\gamma)$. However, they share the same region $\{(\beta,\gamma)\colon \beta+\gamma \leqslant 1\}$ in which f is univalent. □

3. Univalence of an integral operator of $L(\beta, \gamma)$

The univalence of some integral operators for univalent families like S, L, S^c, and in particular the univalence of

$$F_\alpha(z) = F_\alpha(f)(z) = \int_0^z \big(f'(t)\big)^\alpha \, dt, \quad \alpha \in \mathbb{R}(\mathbb{C}), \tag{14}$$

was studied in several papers. Here we solve the problem of univalence of (14) for $f \in L(\beta, \gamma)$ and $\alpha \in \mathbb{R}$ by applying the method from [11]. According to Pfaltzgraff's theorem [8] the integral in (14) is univalent for $f \in L(\beta, \gamma)$ if

$$|\alpha| \leqslant \frac{1}{2(\beta + \gamma + 1)}, \quad \alpha \in \mathbb{C}.$$

However, for $\alpha \in \mathbb{R}$ the above region can be extended considerably and will be sharp.

We will use the following result.

Lemma 3. *The minimal invariant family containing the set* $\{F_\alpha(z)\colon\ f \in L(\beta, \gamma)\}$ *is the set of functions*

$$G_\alpha(z) = G_\alpha(f)(z) = \int_0^z \frac{(f'(t))^\alpha}{(1 - \xi z)^{2-2\alpha}} \, dt, \quad \xi \in \mathbb{D} \setminus \{0\},\ \alpha \in \mathbb{R}. \tag{15}$$

The order of the family $\{G_\alpha(f)\}$ *is equal to*

$$|\alpha|(\beta + \gamma + 1) + |1 - \alpha|.$$

Proof. The first part of Lemma 3 holds for any invariant family and was proved in [11]. To calculate the order, notice that by Lemma 2

$$\sup_{f \in L(\beta,\gamma)} \frac{1}{2}\big|G''_\alpha(0)\big| = \sup_{f \in L(\beta,\gamma)} \big|\alpha a_2 + (1 - \alpha)\xi\big| = |\alpha|(\beta + \gamma + 1) + |1 - \alpha|. \qquad \square$$

Theorem 5. *Let* $f \in L(\beta, \gamma)$ *and* $\alpha \in \mathbb{R}$. *The integral in* (15) *is univalent in the disk* $|z| \leqslant r_u^\alpha(\beta, \gamma)$, *where*

$$r_u^\alpha(\beta, \gamma) \geqslant \min\left\{1; \tan \frac{\pi}{2[|\alpha|(\beta + \gamma + 1) + |1 - \alpha|]}\right\}. \tag{16}$$

The same conclusion holds for the integral (14).

Proof. From (15) and (8) we obtain

$$\begin{aligned} \big|\arg G'_\alpha(z)\big| &\leqslant |\alpha|\big|\arg f'(z)\big| + 2|1 - \alpha|\big|\arg(1 - \xi z)\big| \\ &\leqslant 2\big\{|\alpha|(\beta + \gamma + 1) + |1 - \alpha|\big\} \arcsin r. \end{aligned} \tag{17}$$

The rest of the proof follows the same line of reasoning as in the proof of Theorem 3. $\square$

Using Lemma 3 and Lemma A with $\tau = |\alpha|(\beta + \gamma + 1) + |1 - \alpha|$ and the bound given in (17), we can find region for $\alpha \in \mathbb{R}$, when f is close-to-convex univalent in $\mathbb{D}$ which will strengthen and make sharp the conclusion given in (16). Namely, we have the following theorem by Lemma A.

Theorem 6. *If $f \in L(\beta, \gamma)$ and $\alpha \in \mathbb{R}$, then the integral in* (15) *is univalent and close-to-convex for all $\alpha \in \mathbb{R}$ such that $|\alpha|(\beta + \gamma + 1) + |1 - \alpha| \leqslant 2$. If $|\alpha|(\beta + \gamma + 1) + |1 - \alpha| > 2$ then the radius of close-to-convexity of* (15) *is the unique solution of Eq.* (12) *with $\tau = |\alpha|(\beta + \gamma + 1) + |1 - \alpha|$. The same conclusion holds for the integral in* (14) *with $f \in L(\beta, \gamma)$ and this is sharp.*

Corollary 4. *If $f \in L(\beta, \gamma)$, then the integral in* (14) *is univalent for*

$$\alpha \in \left[\frac{-1}{\beta + \gamma + 2}, \frac{3}{\beta + \gamma + 2}\right] \quad \text{if } \beta + \gamma \leqslant 1$$

and

$$\alpha \in \left[\frac{-1}{\beta + \gamma + 2}, \frac{1}{\beta + \gamma}\right] \quad \text{if } \beta + \gamma \geqslant 1.$$

The result is sharp.

Putting $\beta = \gamma = 0$ and $\beta = 1$, $\gamma = 0$ we get the following results proved in [7] by different methods.

Corollary 5. *If $f \in S^c$, then the integral in* (14) *is univalent and close-to-convex for all $\alpha \in [-1/2, 3/2]$ and this is sharp.*

Corollary 6. *If $f \in L$, then the integral in* (14) *is univalent and close-to-convex for all $\alpha \in [-1/3, 1]$ and this is sharp.*

Remark. Does the class $L(\beta, \gamma)$ and in particular $L(1, 1)$ or $L(1/2, 1/2)$ have any interesting geometric interpretation (like accessibility of $f(\mathbb{D})$ by angles from the complement of $f(\mathbb{D})$)?

References

[1] D.M. Campbell, M.R. Ziegler, The argument of the derivative of linear-invariant families of finite order and the radius of close-to-convexity, Ann. Univ. Mariae Curie-Skłodowska Sect. A 28 (1974) 5–22.

[2] A.W. Goodman, On close-to-convex functions of higher order, Ann. Univ. Sci. Budapest. Eötvös Sect. Math. 15 (1972) 17–30.

[3] A.W. Goodman, Univalent Functions, vols. I, II, Mariner Publ. Co., 1983.

[4] W. Hengartner, G. Schober, Analytic functions close-to-mappings convex in one direction, Proc. Amer. Math. Soc. 28 (1971) 519–524.

[5] W. Koepf, Close-to-convex functions and linear invariant families, Ann. Acad. Sci. Fenn. Ser. I Math. 8 (1983) 349–355.

[6] J. Krzyż, Some remarks on close-to-convex functions, Bull. Acad. Polon. Sci. Ser. Sci. Math. Astron. Phys. 10 (1962) 201–204.
[7] E.P. Merkes, D.J. Wright, On the univalence of certain integral, Proc. Amer. Math. Soc. 27 (1971) 97–100.
[8] J.A. Pfaltzgraff, Univalence of the integral of $f'(z)^{\lambda}$, Bull. London Math. Soc. 7 (1975) 254–256.
[9] C. Pommerenke, Linear-invariante Familien analytischer Funktionen I, Math. Ann. 155 (1964) 108–154.
[10] C. Pommerenke, On close-to-convex analytic functions, Trans. Amer. Math. Soc. 114 (1964) 176–186.
[11] D.J. Prokhorov, J. Szynal, On the radius of univalence for the integral $f'(z)^{\alpha}$, Ann. Univ. Mariae Curie-Skłodowska Sect. A 33 (1979) 157–163.
[12] J. Szynal, J. Waniurski, Some problems for linearly invariant families, Ann. Univ. Mariae Curie-Skłodowska Sect. A 30 (1976) 91–102.