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TYPICALLY REAL HARMONIC FUNCTIONS

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ABSTRACT. We consider a class $\mathcal{T}_H^O$ of typically real harmonic functions on the unit disk that contains the class of normalized analytic and typically real functions. We also obtain some partial results about the region of univalence for this class.

1. Introduction. A planar harmonic mapping is a complex-valued function $f = u + iv$, for which both u and v are real harmonic. If G is simply connected, then f can be written as $f = h + \overline{g}$, where h and g are analytic on G. The reader is referred to [**4**] for many interesting results on planar harmonic mappings. Throughout this paper we will discuss harmonic functions on the unit disk $\mathbf{D}$. Analogously to the classical family S of normalized analytic schlicht functions and its subfamilies K of convex mappings and C of close-to-convex mappings, Clunie and Sheil-Small [**3**] introduced the class $S_H^O = \{f : \mathbf{D} \to \mathbf{C} \mid f \text{ is harmonic, univalent with } f(0) = h(0) = 0, f_z(0) = h'(0) = 1, f_{\bar{z}}(0) = g'(0) = 0\}$ and its corresponding subclasses K_H^O and C_H^O. Note that $S \subset S_H^O$, $K \subset K_H^O$ and $C \subset C_H^O$. Another well-known class of analytic functions in $\mathbf{D}$ is the family, T, of typically real functions that have the normalization $f(z) = z + a_2 z^2 + \cdots$ and are real if and only if z is real. Clunie and Sheil-Small introduced the family of harmonic typically real functions f for which $f(z)$ is real if and only if z is real. Then they proposed the following class of harmonic typically real functions.

Definition [**3**]. Let T_H be the class of typically real harmonic functions $f = h + \overline{g}$ such that $|g'(z)| < |h'(z)|$ for all $z \in \mathbf{D}$, $f(0) = 0$, $|h'(0)| = 1$, and $f(r) > 0$ for $0 < r < 1$. Let T_H^O be the subclass of T_H with $g'(0) = 0$.

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Note that T_H is normal and T_H^O is compact. Besides Clunie and Sheil-Small, several other authors have investigated harmonic real functions (see [**2, 18**]).

The condition that $|h'(z)| > |g'(z)|$ means that $f = h + \overline{g}$ must be locally univalent and sense-preserving (see Lewy [**11**]). However, not all analytic typically real functions are locally univalent. Thus, a problem with this definition is that it prevents the family of analytic typically real functions from being a subset of their family of harmonic typically real functions, that is, $T \not\subset T_H^O$.

To resolve this problem and allow all analytic typically real functions to also be harmonic typically real functions, we offer a slightly different definition for a family of harmonic typically real functions, $\mathcal{T}_H^O$. In particular, we reduce the requirement that the harmonic functions must be locally univalent. This means that the standard results for harmonic locally univalent functions must be reconsidered for this family. We therefore show that for the family $\mathcal{T}_H^O$ Clunie and Sheil-Small's shearing technique still holds. Also, as in the case for the family of analytic typically real functions we investigate the region of univalency for the harmonic family and provide several conjectures for $\mathcal{T}_H^O$.

2. The class $\mathcal{T}_H^O$. For the harmonic function $f = h + \overline{g}$, let ω be given by $g'(z) = \omega(z)h'(z)$. We say that f is sense-preserving at a point z_0 if $h'(z) \not\equiv 0$ in some neighborhood of z_0 and ω is analytic at z_0 with $|\omega(z_0)| < 1$. If f is sense-preserving at z_0, then either the Jacobian $J_f(z_0) = |h'(z_0)|^2 - |g'(z_0)|^2 > 0$ or $h'(z_0) = 0$ for an isolated point z_0 as was mentioned by Duren, Hengartner and Laugesen [**5**]. That is, z_0 is a removable singularity of the meromorphic functions ω and $|\omega(z_0)| < 1$. We say f is sense-preserving in $\mathbf{D}$ if f is sense-preserving at all $z \in \mathbf{D}$. By requiring the harmonic function f to be sense-preserving, we retain some important properties exhibited by analytic functions, such as the open mapping property, the argument principle, and zeros being isolated (see [**5**]). We note that the following harmonic typically real functions

$$f_1(z) = z - \overline{z} \text{ and } f_2(z) = 2(1+i)z + iz^2 + \overline{2(-1+i)z + iz^2}.$$

are not sense-preserving, and they do not have the properties mentioned above.

Thus, we give the following definition.

Definition 1. Let $\mathcal{T}_H$ be the class of typically real harmonic functions, f, such that f is a sense-preserving harmonic function, $f(z)$ is real if and only if z is real, $f(0) = 0$, $|h'(0)| = 1$, and $f(r) > 0$ for $0 < r < 1$. Let $\mathcal{T}_H^O$ be the subclass of $\mathcal{T}_H$ with $g'(0) = 0$.

Also, notice that $T \cup T_H^O \subset \mathcal{T}_H^O$ and, with this definition as in the analytic case, a harmonic typically real function need not be univalent or even locally univalent.

Theorem 1. *If $f \in \mathcal{T}_H$, then f is strictly increasing on the real interval $(-1, 1)$. Moreover, if $f = h + \overline{g} \in \mathcal{T}_H^O$ and $g(0) = 0$, then $h'(0) = f_z(0) = 1$.*

Proof. Observe that the derivative f' exists on the interval $(-1, 1)$ and $f' = h' + \overline{g'}$, $\operatorname{Im} h = \operatorname{Im} g$ there. Suppose that there exists a point $x_0 \in (-1, 1)$ such that $f'(x_0) = 0$. This implies that $J_f(x_0) = 0$. As we know this can only occur if $h'(x_0) = 0 = g'(x_0)$ with the order of zero of g' greater than or equal to the order of h'. Hence, $(h - g)'(x_0) = 0$ contrary to the fact that $h - g$ is a typically real analytic function and such functions are known to be univalent in the lens domain bounded by the circles $|z \pm i| = \sqrt{2}$ [**6, 12**]. $\square$

Now, we note that the basic shearing theorem by Clunie and Sheil-Small [**3**, Theorem 5.3] still holds when local univalence is omitted. That is, we have the following version.

Theorem 2. *Let $f = h + \overline{g}$ be sense-preserving harmonic on $\mathbf{D}$. Then f is univalent and convex in the horizontal direction on $\mathbf{D}$ if and only if $h - g$ has the same properties.*

Proof. We only need to prove the reverse direction. So assume that $F = h - g$ is univalent and convex in the horizontal direction. Consider

$$\begin{aligned} G(w) &= f(F^{-1}(w)) = h(F^{-1}(w)) + \overline{g(F^{-1}(w))} \\ &= w + 2\operatorname{Re}\left\{g(F^{-1}(w))\right\}. \end{aligned}$$

If G is locally univalent in $\Omega = F(\mathbf{D})$, then we can apply the same approach as in Clunie and Sheil-Small's proof. In particular, by their lemma ([**3**, page 13]), G is univalent in Ω and has an image that is convex in the horizontal direction, and consequently, so is f. Therefore, we only need to show that G is locally univalent. To do this, consider the Jacobian of G:

$$\begin{aligned} J_G &= \left|\frac{d}{dw} h\circ F^{-1}\right|^2 - \left|\frac{d}{dw} g\circ F^{-1}\right|^2 \\ &= (|h'\circ F^{-1}|^2 - |g'\circ F^{-1}|^2)\cdot |(F^{-1})'|^2 \\ &= J_{f\circ F^{-1}}\cdot |(F^{-1})'|^2. \end{aligned}$$

Now suppose there exists a point $z_0 \in \mathbf{D}$ such that $J_G(F(z_0)) = 0$. Since $(F^{-1})'(w) \neq 0$ on $F(\mathbf{D})$, we have that $|h'(z_0)| = |g'(z_0)|$. As mentioned above, this is only possible when $h'(z_0) = 0 = g'(z_0)$ which contradicts the assumption that $F = h - g$ is univalent. □

Next, we give a representation formula and extreme points for functions in the class $\mathcal{T}_H^O$.

Let $\mathcal{P}$ denote the class of all functions of the form $p(z) = 1 + p_1 z + p_2 z^2 + \cdots$ that are analytic in $\mathbf{D}$ and such that $\operatorname{Re} p(z) > 0$ for $z \in \mathbf{D}$. By the well-known Herglotz representation formula $p \in \mathcal{P}$ if and only if there exists a unique probability measure μ on $\partial\mathbf{D}$ such that

$$p(z) = \int_{\partial\mathbf{D}} \mathbf{p}_\eta(z)\, d\mu(\eta), \quad z \in \mathbf{D}, \tag{2.1}$$

where

$$\mathbf{p}_\eta(z) = (1+\eta z)/(1-\eta z). \tag{2.2}$$

Moreover, if $p \in \mathcal{P}$ has real Taylor coefficients, then

$$p(z) = \int_{-1}^{1} \frac{1-z^2}{1-2tz+z^2}\, d\nu(t), \quad z \in \mathbf{D}$$

with the unique probability measure ν on the segment $[-1,1]$. This in turn implies that for an analytic function F in the class T we have the following Robertson representation formula

$$F(z) = \int_{-1}^{1} \frac{z\, d\nu(t)}{1-2tz+z^2}, \quad z \in \mathbf{D}, \tag{2.3}$$

where ν is as above. Recall Rogosinski's result [**14**] that a (normalized) typically real function f can be written as $f(z) = (z/1-z^2)p(z)$ for some $p \in \mathcal{P}$. The set of extreme points of the class T consists of the functions

$$z \longmapsto \mathbf{q}_t(z) = \frac{z}{1-2tz+z^2}, \quad -1 \le t \le 1. \tag{2.4}$$

The shearing construction can be applied to the class $\mathcal{T}_H^O$. Consequently, we see that every $f = h + \overline{g} \in \mathcal{T}_H^O$ can be written in the form

$$f(z) = \operatorname{Re} \int_0^z p(\zeta)F'(\zeta)\, d\zeta + i \operatorname{Im} F(z) = k(z,p,F), \tag{2.5}$$

where $F = h - g \in T$ and $p = (1+\omega)/(1-\omega) \in \mathcal{P}$ with $\omega = g'/h'$, where removable singularities are admitted. Also, given $p \in \mathcal{P}$ and $F \in T$, the function f defined by (2.5) is in $\mathcal{T}_H^O$ and $k(\cdot,p,F) = h + \overline{g}$, with

$$h(z) = \frac{1}{2}\int_0^z (p(\zeta)+1)F'(\zeta)\, d\zeta = z + a_2 z^2 + \cdots,$$
$$g(z) = \frac{1}{2}\int_0^z (p(\zeta)-1)F'(\zeta)\, d\zeta = b_2 z^2 + b_3 z^3 + \cdots.$$

Note also that the function $f = k(\cdot,p,F)$ is locally univalent if and only if F is a locally univalent function. This is a consequence of the equality

$$J_f(z) = |F'(z)|^2 \operatorname{Re} p(z), \quad z \in \mathbf{D}. \tag{2.6}$$

Furthermore, we have

Theorem 3. *The class $\mathcal{T}_H^O$ is compact (in the topology of uniform convergence on the compact subsets of $\mathbf{D}$) and the set* ext $(\mathcal{T}_H^O)$ *of its extreme points consists of the functions $k(\cdot, \mathbf{p}_\eta, \mathbf{q}_t)$, where $\mathbf{p}_\eta$ and $\mathbf{q}_t$ are given by* (2.2) *and* (2.4), *respectively. The class $\mathcal{T}_H^O$ is not convex.*

Proof. Compactness of the class $\mathcal{T}_H^O$ follows immediately from compactness of both classes T and $\mathcal{P}$. Assume that $f = k(\cdot,p,F) \in$ ext $(\mathcal{T}_H^O)$ and there is a $0 < \lambda < 1$ such that either

(i) $p = (1-\lambda)p_1 + \lambda p_2$, with $p_1, p_2 \in \mathcal{P}$, $p_1 \neq p_2$,

or

(ii) $F = (1-\lambda)F_1 + \lambda F_2$, with $F_1, F_2 \in T$, $F_1 \neq F_2$.

Then

$$f = (1-\lambda)f_1 + \lambda f_2,$$

where, in case (i):

$$f_j = k(\cdot, p_j, F) \quad \text{with} \quad (f_1)_z - (f_2)_z = (p_1 - p_2)F'/2,$$

which implies $f_1 \neq f_2$, a contradiction; and in case (ii):

$$f_j = k(\cdot, p, F_j) \quad \text{with} \quad (f_1)_z - (f_2)_z = (p+1)\frac{F_1' - F_2'}{2},$$

a contradiction again. Thus, by the Herglotz and Robertson formulas, we get $\operatorname{ext}(\mathcal{T}_H^O) \subset \{k(\cdot, \mathbf{p}_\eta, \mathbf{q}_t), |\eta| = 1, -1 \leq t \leq 1\}$. Now if

$$f = k(\cdot, \mathbf{p}_\eta, \mathbf{q}_t) = (1-\lambda)f_1 + \lambda f_2 = (1-\lambda)k(\cdot, p_1, F_1) + \lambda k(\cdot, p_2, F_2),$$

then

$$\mathbf{q}_t' = f_z - \overline{f_{\bar z}} = (1-\lambda)F_1' \lambda F_2',$$

which gives $\mathbf{q}_t = F_1 = F_2$; and

$$\mathbf{p}_\eta \mathbf{q}_t' = f_z + \overline{f_{\bar z}} = (1-\lambda)p_1 F_1' + \lambda p_2 F_2' = ((1-\lambda)p_1 + \lambda p_2)\mathbf{q}_t',$$

which implies $p_1 = p_2 = \mathbf{p}_\eta$. Consequently, $f_1 = f_2$ and $f \in \operatorname{ext}(\mathcal{T}_H^O)$.

Finally, we show that the class $\mathcal{T}_H^O$ is not convex. More exactly, we show that for arbitrary $\xi, \eta \in \partial\mathbf{D}$, $s, t \in [-1, 1]$, $\xi \neq \eta$, $s \neq t$ and $0 < \lambda < 1$,

$$f = (1-\lambda)k(\cdot, \mathbf{p}_\xi, \mathbf{q}_s) + \lambda k(\cdot, \mathbf{p}_\eta, \mathbf{q}_t) \notin \mathcal{T}_H^O.$$

Suppose, contrary to our claim, that $f \in \mathcal{T}_H^O$. Then there exist $p \in \mathcal{P}$ and $F \in T$ such that $f = k(\cdot, p, F)$ and

$$F' = f_z - \overline{f_{\bar z}} = (1-\lambda)\mathbf{q}_s' + \lambda \mathbf{q}_t'.$$

This implies that $F = (1-\lambda)\mathbf{q}_s + \lambda\mathbf{q}_t$. Moreover, we have

$$pF' = f_z + \overline{f_{\bar z}} = (1-\lambda)\mathbf{p}_\xi\mathbf{q}_s' + \lambda\mathbf{p}_\eta\mathbf{q}_t'.$$

Since the image of $\mathbf{D}$ under an analytic branch of $\sqrt{\mathbf{q}_s'/\mathbf{q}_t'}$ contains the upper and lower half planes, there exists an $a \in \mathbf{D}\setminus\{0\}$ such that $\mathbf{q}_s'(a)/\mathbf{q}_t'(a) = -\lambda/(1-\lambda)$. Hence $F'(a) = 0$ and

$$\begin{aligned} p(a)F'(a) &= (1-\lambda)\mathbf{p}_\xi(a)\mathbf{q}_s'(a) + \lambda\mathbf{p}_\eta(a)\mathbf{q}_t'(a) \\ &= \lambda\mathbf{q}_t'(a)(\mathbf{p}_\eta(a) - \mathbf{p}_\xi(a)) \neq 0, \end{aligned}$$

a contradiction. □

As a corollary to Theorem 3 we get the same sharp coefficient estimates for the class $\mathcal{T}_H$ and $\mathcal{T}_H^O$ as were found by Clunie and Sheil-Small [**3**] for $T_H \subset \mathcal{T}_H$ and $T_H^O \subset \mathcal{T}_H^O$.

3. Region of univalence. For $z_0 \in \mathbf{C}$ and positive r, let $D(z_0; r)$ denote the open disk centered at z_0 with radius r. We mentioned in the Introduction that an analytic function $f \in T$ need not be univalent in $\mathbf{D}$, but it is univalent in the lens domain

$$L = D(-i;\sqrt{2}) \cap D(i;\sqrt{2}).$$

The result was obtained by Goluzin [**6**] and by Merkes [**12**] independently. They also noted that this region of univalence for class T cannot be extended, because for each $z_0 \in \partial L \cap \mathbf{D}$ there exists a parameter $t_0 \in (0,1)$ such that $f_{t_0}'(z_0) = 0$, where

$$f_t(z) = \frac{tz}{(1-z)^2} + \frac{(1-t)z}{(1+z)^2}. \tag{3.1}$$

This can also be shown by noting that

$$\partial L \cap \mathbf{D} = \left\{ z \in \mathbf{D} : \left(\frac{1+z}{1-z}\right)^4 < 0 \right\}$$

and

$$f_t'(z) = \left(\left(\frac{1+z}{1-z}\right)^4 + \frac{1-t}{t} \right) \frac{t(1-z)}{(1+z)^3}.$$

Let us observe that actually for each $z_0 \in \mathbf{D} \setminus L$ there exist $t_0 \in (0,1)$ and $R \in (\sqrt{2}-1, 1]$ such that $Rz_0 \in \partial L$ and $f'_{t_0,R}(z_0) = 0$, where $f_{t,R}(z) = f_t(Rz)/R$ and f_t is defined by (3.1). Note that the function $f_{t,R}$ as a convex combination of univalent functions with real coefficients is in class T.

As in the analytic case, a harmonic typically real function need not be univalent. Therefore, Złotkiewicz posed the problem of determining the region of univalence for harmonic typically real functions. Before we give a partial answer to this question we present a simple proof of the Goluzin-Merkes result for analytic typically real functions (based on Merkes's idea). To this end, note first that the function

$$\zeta = \psi(z) = \frac{2z}{1+z^2} \tag{3.2}$$

maps conformally the disk $\mathbf{D}$ onto the two-slit plane cut along the real intervals $(-\infty, -1]$ and $[1, \infty)$. Since the function ψ is typically real, there is a one-to-one correspondence between the class T and the class of normalized and typically real functions in $\Omega = \mathbf{C} \setminus ((-\infty, -1] \cup [1, \infty))$. Moreover, using the Robertson formula, we get the following formula for a typically real function F in Ω with normalization $F(0) = F'(0) - 1 = 0$ and the one-to-one correspondence:

$$F(\zeta) = \int_{-1}^{1} \frac{\zeta d\nu(t)}{1-t\zeta}, \quad f = \frac{1}{2} F \circ \psi \in T, \tag{3.3}$$

where ν is a probability measure on $[-1, 1]$. It has been observed in [**15, 16**] that F restricted to disk $\mathbf{D}$ is univalent. Consequently, any function $f \in T$ is univalent on the preimage of the unit disk under the function ψ given by (3.2), which is the lens domain L.

In 1936 Robertson observed that an analytic function F with real coefficients is univalent and convex in the vertical direction if and only if the function $z \mapsto zF'(z)$ is typically real (see [**8**, page 206]). Hence the functions given by (3.3) are convex in the direction of the imaginary axis (see also [**12, 13**]). Therefore, the sets $f(L)$, $f \in T$, are convex in the vertical direction. Moreover, we will show the following interesting property of class T.

Proposition. *For a $z \in \partial L \cap \mathbf{D}$ there exists a unique $f \in T$ for which $f'(z) = 0$.*

Proof. By (3.3) it is enough to consider the equation

$$\begin{aligned}0 = F'(e^{i\alpha}) &= \int_{1}^{1} \frac{d\nu(t)}{(1-te^{i\alpha})^2} \\ &= \int_{-1}^{1} \frac{1-t^2}{|1-te^{i\alpha}|^4} d\nu(t) \\ &\quad - 2\cos\alpha \int_{-1}^{1} \frac{t(1-t\cos\alpha)}{|1-te^{i\alpha}|^4} d\nu(t) \\ &\quad + 2i\sin\alpha \int_{-1}^{1} \frac{t(1-t\cos\alpha)}{|1-te^{i\alpha}|^4} d\nu(t),\end{aligned}$$

where $0 < \alpha < \pi$. It then follows

(i)

$$\int_{-1}^{1} \frac{t(1-t\cos\alpha)}{|1-te^{i\alpha}|^4} d\nu(t) = 0$$

and, consequently,

(ii)

$$\int_{-1}^{1} \frac{1-t^2}{|1-te^{i\alpha}|^4} d\nu(t) = 0.$$

From equality (ii) we get $\nu = (1-\lambda)\delta_{-1} + \lambda\delta_1$ for some $\lambda \in [0,1]$, Finally, equality (i) gives $\lambda = \sin^2(\alpha/2)$. □

Corollary. *Let $f \in T$. Then either f is univalent on $\overline{L} \setminus \{-1,1\}$ or there is a unique $t \in (0,1)$ such that $f = f_t$, where f_t is given by* (3.1). *Moreover, $f_t(L) = \mathbf{C} \setminus \{(1-2t)/4 + i\lambda : \lambda \in \mathbf{R},\ |\lambda| \geq \sqrt{t(1-t)}/2\}$.*

Proof. Clearly f is analytic on $\gamma = \partial L \setminus \{-1,1\}$ and $\operatorname{Re} f(z)$ changes monotonically. It is sufficient to show that $\operatorname{Re} f(z)$ is not constant on any arc $\gamma_0 \subset \gamma$ or $f = f_t$ for some $t \in (0,1)$. If f is constant on an arc $\gamma_0 \subset \gamma$ lying in the upper half-plane, then the function given by

$$g(z) = f(z) + \overline{f\left(-i + \frac{2}{\overline{z} - i}\right)}$$

is analytic on a neighborhood of γ_0 and $g(z) = 2\operatorname{Re} f(z)$ on γ_0. So, $g(z) = \text{const}$ on γ_0 and, consequently, g is a constant function. This means that $\operatorname{Re} f$ is constant on γ. Consequently, the boundary value of f at 1 and -1 is equal to ∞, so there is a $z \in \partial L \cap \mathbf{D}$ such that $f'(z) = 0 = F'(\psi(z)) = 0$. Hence by the Proposition $f = f_t$, where $t = (1 - \operatorname{Re}\psi(z))/2$. □

We also note that the radius of starlikeness for class T is $\sqrt{2} - 1$ [**9**]. Moreover, every $f \in T$ is univalent on $D(0; \sqrt{2} - 1)$ and the curve $f(\partial D(0; \sqrt{2} - 1))$ is strictly starlike with respect to the origin. Indeed, if we put $g = zf'/f$, then the function defined by $G(z) = g(z) + g((3 - 2\sqrt{2})/\overline{z})$ is analytic on a neighborhood of the circle $\partial D(0; \sqrt{2} - 1)$. Hence for $|z| = \sqrt{2} - 1$ we have $G(z) = 2\operatorname{Re}\{zf'(z)/f(z)\} > 0$, except for a finite number of points at which it vanishes.

We have already shown that every harmonic typically real function in the sense of Definition 1 is strictly monotonic on the interval $(-1, 1)$. Moreover, we have the following

Theorem 4. *For each function f in $\mathcal{T}_H^O$ there exists an open set V, $(-1, 1) \subset V \subset \mathbf{D}$, such that f is univalent on V.*

Proof. Let $f = k(\cdot, p, F)$ with $p \in \mathcal{P}$ and $F \in T$. We first show that, for a compact interval $[a, b] \subset (-1, 1)$, there is an open set U containing $[a, b]$ and such that f is univalent on U. Clearly, $[F(a), F(b)] \subset F(L)$, where L is the lens domain defined above. Since F(L) is an open set, there exist $\delta > 0$ and $c > 0$ such that $(F(a) - \delta, F(b) + \delta) \times (-c, c) \subset F(L)$. Let U be the preimage of set $(F(a) - \delta, F(b) + \delta) \times (-c, c)$ under F. Then

$$U = U(a, b, c, \delta) = \bigcup_{-c<d<c} z_d((F(a) - \delta, F(b) + \delta)),$$

where $z_d(t) = F^{-1}(t + id)$, $F(a) - \delta < t < F(b) + \delta$. Now note that, since F is univalent on L, the curves z_d, $-c < d < c$ are disjoint and

$$\frac{d}{dt}\operatorname{Re} f(z_d(t)) = \operatorname{Re}\{p(z_d(t))F'(z_d(t))z_d'(t)\} = \operatorname{Re} p(z_d(t)) > 0.$$

This and the fact that $\operatorname{Im} f = \operatorname{Im} F$ imply the univalence of f on U.

Let $\{a_n\}$ be a strictly decreasing sequence of negative numbers converging to -1 and $\{b_n\}$ a strictly increasing sequence of positive numbers converging to 1. Then, for each positive integer n, we can find $\delta_n > 0$, $c_n > 0$ and the open set $U_n = U(a_n, b_n, c_n, \delta_n)$ such that f is univalent on U_n. Now set $\delta'_n = \min\{F(a_n) - F(a_{n+1}), F(b_{n+1}) - F(b_n), \delta_n\}$ and $c'_1 = c_1$, $c'_{n+1} = \min\{c'_n, c_{n+1}\}$, $n = 1, 2, \ldots$, and define

$$V = \bigcup_{n=1}^{\infty} U(a_n, b_n, c'_n, \delta'_n).$$

Clearly, $(-1,1) \subset V$. Moreover, f is univalent on V. To see this suppose that $f(z) = f(w)$ and $z \in U(a_n, b_n, c'_n, \delta'_n)$, $w \in U(a_{n+k}, b_{n+k}, c'_{n+k}, \delta'_{n+k})$, $k \geq 1$. Since $\operatorname{Im} F = \operatorname{Im} f$, we get $z \in U(a_{n+k}, b_{n+k}, c'_{n+k}, \delta'_{n+k})$ and consequently, $z = w$. □

Remark 1. It is clear that, for every continuous mapping f of a neighborhood of interval $(-1,1)$ into $\mathbf{C}$ such that $f((-1,1)) \subset \mathbf{R}$ and f is a local homeomorphism of $(-1,1)$, there is a domain Ω and a simply connected domain G such that $(-1,1) \subset \Omega$ and f is a local homeomorphism of Ω onto G. If the pair (Ω, f) is an unlimited covering space of domain G, then by the Monodromy theorem f is a homeomorphism of Ω onto G [**1**]. In general, such a situation is rare. The example below shows that f may be infinite-valent on Ω, so that the typically real property in the proof of Theorem 4 seems to be essential.

Example. Let $u(z) \equiv (4z)/(1+z)^2$, $f(\xi) \equiv \xi e^{-\xi}$. It is clear that the function $f \circ u$ is locally univalent on $\mathbf{D}$. By the Great Picard theorem, $f \circ u(\mathbf{D}) = \mathbf{C}$ and every value $w \in \mathbf{C} \setminus \{0\}$ is assumed by $f \circ u$ at infinitely many points of each set $\mathbf{D} \cap \{z : |z+1| < \delta\}$, where $0 < \delta < 2$.

Next, we show that the region, L, of univalency for class T is not the region of univalency for class $\mathcal{T}_H^O$.

Theorem 5. *There exist functions $f \in \mathcal{T}_H^O$ that are not univalent on L.*

Proof. Put

$$F(z) = f_{1/2}(z) = \frac{1}{2}\left(\frac{z}{(1+z)^2} + \frac{z}{(1-z)^2}\right), \quad z \in \mathbf{D},$$

and define $f \in \mathcal{T}_H^O$ by the formula

$$f(z) = \operatorname{Re} \int_0^z \frac{1+\zeta}{1-\zeta} F'(\zeta)\, d\zeta + i \operatorname{Im} F(z).$$

Suppose that f is univalent on L. Then the function $g = f \circ \psi^{-1}$, where ψ is given by (3.2), is univalent on $\mathbf{D}$. A calculation gives

$$g(w) = \operatorname{Re}\left(\frac{1+w}{12(1-w)}\sqrt{\frac{1+w}{1-w}} - \frac{1}{4}\sqrt{\frac{1-w}{1+w}} + \frac{1}{6}\right) + \frac{i}{2}\operatorname{Im}\left(\frac{w}{1-w^2}\right),$$

where we assume that $\sqrt{1} = 1$. Now, note that, for $0 < \alpha < \pi/2$,

$$\operatorname{Im}\left(g(ie^{-i\alpha}) - g(ie^{i\alpha})\right) = 0.$$

Moreover, we have

$$\begin{aligned}\operatorname{Re}\left(g(ie^{-i\alpha}) - g(ie^{i\alpha})\right) &= \frac{1}{12\sqrt{2}}\left(\cot^{3/2}\left(\frac{\pi}{4} + \frac{\alpha}{2}\right) - \cot^{3/2}\left(\frac{\pi}{4} - \frac{\alpha}{2}\right)\right) \\ &\quad - \frac{1}{4\sqrt{2}}\left(\cot^{1/2}\left(\frac{\pi}{4} + \frac{\alpha}{2}\right) - \cot^{1/2}\left(\frac{\pi}{4} - \frac{\alpha}{2}\right)\right) \\ &= \frac{C}{12\sqrt{2}}\left(\cot^{1/2}\left(\frac{\pi}{4} + \frac{\alpha}{2}\right) - \cot^{1/2}\left(\frac{\pi}{4} - \frac{\alpha}{2}\right)\right),\end{aligned}$$

where

$$C = \cot\left(\frac{\pi}{4} + \frac{\alpha}{2}\right) - 2 + \frac{1}{\cot((\pi/4) + (\alpha/2))} > 0.$$

This means that, for $0 < \alpha < \pi/2$,

$$\operatorname{Re}\left(g(ie^{-i\alpha}) - g(ie^{i\alpha})\right) < 0.$$

To get a contradiction consider function m defined by

$$m(r,\alpha) = \mathrm{Re}\,\left(g(rie^{-i\alpha}) - g(rie^{i\alpha})\right).$$

The function m is uniformly continuous on the rectangle $[0,1]\times[0,\pi/4]$ and $m(1,\alpha) < 0$ for $0 < \alpha < \pi/4$. On the other hand,

$$m(r,\alpha) = r(\sin\alpha + o(1)) \quad \text{as } r \to 0^+.$$

Consequently, for every $\alpha \in (0,\pi/4)$ there is an $r_\alpha \in (0,1)$ such that $m(r_\alpha,\alpha) = 0$. This means that $g(r_\alpha ie^{-i\alpha}) = g(r_\alpha ie^{i\alpha})$, a contradiction. □

Theorem 6. *Every function $f \in \mathcal{T}_H^O$ is univalent in any of the following domains*:

(a) *the disk* $D(0;\sqrt{6}-\sqrt{5})$,

(b) $\{z \in \mathbf{D} : |(2z)/(1+z^2)| < \sqrt{2}-1\}$.

Proof. It follows from (2.6) that every $f = h + \overline{g} \in \mathcal{T}_H^O$ is locally univalent on lens domain L. Moreover, by the results in [**17**], $F = h - g$ is convex on $D(0;\sqrt{6}-\sqrt{5})$. Thus, by the shearing theorem of Clunie and Sheil-Small, f is univalent on $D(0;\sqrt{6}-\sqrt{5})$. Note also that it has been shown by Koczan [**10**] that for the class T the radius of convexity in the horizontal direction is exactly $\sqrt{6}-\sqrt{5}$. Now we observe that a function $f \in \mathcal{T}_H^O$ is univalent on the given region in (b) if and only if function $f \circ \psi$, where ψ is given by (3.2), is univalent on the disk $D(0;\sqrt{2}-1)$. The last follows from the fact that an analytic function F given by (3.3) maps the disk $D(0;\sqrt{2}-1)$ onto a convex domain (see [**13**, page 292]) and from the shearing theorem of Clunie and Sheil-Small. □

Clearly, class T_H of typically real harmonic functions introduced by Clunie and Sheil-Small contains locally univalent functions from class T. It would be interesting to find the region of univalence for locally univalent functions that are in T. The following example of the function $G \in T$ that is locally univalent has been described in [**7**]:

$$G(z) = \frac{1}{\pi}\tan\left(\frac{\pi z}{1+z^2}\right), \quad z \in \mathbf{D}.$$

We note that G is univalent in the region $S = \{z \in \mathbf{D} : |\mathrm{Re}((\pi z)/(1+z^2))| < \pi/2\}$ which contains the disk $D(0;1/\sqrt{3})$. Indeed, for $|z| = 1/\sqrt{3}$, we have

$$\left|\mathrm{Re}\,\frac{\pi z}{1+z^2}\right| = \left|\frac{3\pi \mathrm{Re}\,z}{9\mathrm{Re}^2 z + 1}\right| \le \frac{\pi}{2}.$$

Moreover, if $z_0 = (1+i\sqrt{2})/3$, then $z_0, -\overline{z_0} \in \partial D(0;1/\sqrt{3}) \cap \partial S$ and $G(z_0) = G(-\overline{z_0})$. This shows that radius of univalence for the class of locally univalent functions from T is less than or equal to $1/\sqrt{3}$.

Now let r_u^*, respectively r_u, denote the radius of univalence of T_H, respectively $\mathcal{T}_H^O$, that is, the supremum of all $r > 0$ such that every $f \in T_H$, respectively $f \in \mathcal{T}_H^O$, is univalent on $D(0;r)$. Clearly,

$$0.213\cdots = \sqrt{6} - \sqrt{5} \le r_u \le r_u^* \le 1/\sqrt{3} = 0.577\cdots$$

and

$$r_u \le \sqrt{2} - 1 = 0.414\cdots.$$

By examining some computer computations, that will be presented in an upcoming paper, we make the following conjectures.

Conjecture 1. $r_u = \sqrt{2} - 1$.

Conjecture 2. *Every function $f \in \mathcal{T}_H^O$ is univalent on the half-lens* $L \cap \{z : \mathrm{Re}\,z > 0\}$.

We finish the paper with a list of open problems.

(1) Give analytic proofs of Conjectures 1–2.

(2) Prove or disprove that $r_u^* = 1/\sqrt{3}$.

(3) Does there exist an open set U, $(-1,1) \subset U \subset \mathbf{D}$, such that every $f \in \mathcal{T}_H^O$ is univalent on U?

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