Designer Media and Pulses for Optimally Long-Lived and Reversible Energy Storage

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What is free energy?

Gibb's Free energy: "...the **greatest amount of** ... **work** which can be obtained from a given quantity of a certain substance in a given **initial state**–*without increasing its total volume* or allowing heat to pass to or from external bodies, except such as at the close of the processes are left in their initial condition." –J. Willard Gibbs [1]

[1] *A Method of Geometrical Representation of the Thermodynamic Properties of Substances by Means of Surfaces*, Trans. Conn. Acad. II pp.398,399, Dec. 1873

What is a *dynamical/non-equilibrium* free energy?[3]

- Axiom 1: A dynamical free energy is a state function.
- Axiom 2: The Second Law is "**Work** performed **by** a body must be performed **at the expense of at least that much free energy**." [2]
- Result of Axioms–"First Law":

$$\frac{dW_{\textbf{on} \text{ body}}}{dt} = \frac{dF_{\textbf{of} \text{ body}}}{dt} + \frac{dQ_{F \textbf{ loss}}}{dt} \geq \frac{dF_{\textbf{of} \text{ body}}}{dt}$$

- Hindsight Axiom: Both $F_{\textbf{of} \text{ body}}$ *and* $R_{F \textbf{ loss}} := dQ_{F \textbf{ loss}}/dt \geq 0$ are state functions.

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Poynting's Conservation Law+First Law=Dissipation Law

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$$0 \;=\; \nabla \cdot \mathbf{S}(t) + \frac{\partial u_{\text{field}}(t)}{\partial t} + \frac{\partial u_{\text{int}}(t)}{\partial t}$$

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$$u_{\text{int}}(t) \;:=\; \boxed{\int_{-\infty}^{t} E(\tau)\dot{P}(\tau)\,d\tau = W(t) = F(t) + Q(t)}$$

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A Work Representation & a *Bulk* Free Energy: Semi-Infinite Slab $z \in [0, +\infty)$

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The Extreme *Bulk* Free Energy Representations: Semi-Infinite Slab $z \in [0, +\infty)$

$$W[E](t) = c\overbrace{\int_{-\infty}^{t} E^2_{min}\left[E\,(0,*)\right](\tau)d\tau}^{Q_{max}[E](t)} + c\overbrace{\int_{t}^{+\infty} E^2_{min}\left[E_t(0,*)\right](\tau)d\tau}^{F_{min}[E](t)}\,;$$

Standard R.H. Factorization Problem

$$\widehat{E_{min}\,[E]} = N_<\widehat{E}, N_<(\omega) \in \mathcal{A}^+_n, 1/N_<(\omega) \in \mathcal{A}^+_n, N_<(\omega)N_<(-\omega) = n(\omega)$$

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$$\widehat{E_{max}\,[E]} = N_>\widehat{E}, \qquad |N_>(\omega)|^2 = n(\omega)$$

The Extreme *Bulk* Free Energy Representations: Semi-Infinite Slab $z \in [0, +\infty)$

$$W[E](t) = \overbrace{c \int_{-\infty}^{t} E^2_{min}\left[E\,(0,*)\right](\tau) d\tau}^{Q_{max}[E](t)} + \overbrace{c \int_{t}^{+\infty} E^2_{min}\left[E_t(0,*)\right](\tau) d\tau}^{F_{min}[E](t)} ;$$

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$$\widehat{E_{min}\left[E\right]} = N_< \widehat{E}, N_<(\omega) \in \mathcal{A}_n^+, 1/N_<(\omega) \in \mathcal{A}_n^+, N_<(\omega)N_<(-\omega) = n(\omega)$$

$$W[E](t) = \overbrace{c \int_{-\infty}^{t} E^2_{max}\left[E\,(0,*)\right](\tau) d\tau}^{Q_{min}[E](t)} + \overbrace{c \int_{t}^{+\infty} E^2_{max}\left[E_t(0,*)\right](\tau) d\tau}^{F_{max}[E](t)} ;$$

$$\widehat{E_{max}\left[E\right]} = N_> \widehat{E}, \qquad N_>(\omega)N_>(-\omega) = n(\omega)$$

The Extreme *Bulk* Free Energy Representations: Semi-Infinite Slab $z \in [0, +\infty)$

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The Extreme *Bulk* Free Energy Representations: Semi-Infinite Slab $z \in [0, +\infty)$

$$W[E](t) = c \overbrace{\int_{-\infty}^{t} E_{min}^2 \left[E\,(0,*)\right](\tau) d\tau}^{Q_{max}[E](t)} + c \overbrace{\int_{t}^{+\infty} E_{min}^2 \left[E_t(0,*)\right](\tau) d\tau}^{F_{min}[E](t)} ;$$

Standard R.H. Factorization Problem

$$\widehat{E_{min}\,[E]} = N_< \widehat{E}, N_<(\omega) \in \mathcal{A}_n^+, 1/N_<(\omega) \in \mathcal{A}_n^+, N_<(\omega) N_<(-\omega) = n(\omega)$$

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$$\widehat{E_{max}\,[E]} = N_> \widehat{E}, N_>(\omega) \in \mathcal{A}_n^+, 1/N_>(\omega) \in \mathcal{A}_n^-, N_>(\omega) N_>(-\omega) = n(\omega)$$

The Extreme *Bulk* Free Energy Representations: Semi-Infinite Slab $z \in [0, +\infty)$

$$W[E](t) = c \overbrace{\int_{-\infty}^{t} E^2_{min}\left[E\,(0,*)\right](\tau)d\tau}^{Q_{max}[E](t)} + c \overbrace{\int_{t}^{+\infty} E^2_{min}\left[E_t(0,*)\right](\tau)d\tau}^{F_{min}[E](t)}\,;$$

Standard R.H. Factorization Problem

$$\widehat{E_{min}}\,[E] = N_< \widehat{E},\, N_<(\omega) \in \mathcal{A}^+_n,\, 1/N_<(\omega) \in \mathcal{A}^+_n,\, N_<(\omega)N_<(-\omega) = n(\omega)$$

$$W[E](t) = c \overbrace{\int_{-\infty}^{t} E^2_{max}\left[E\,(0,*)\right](\tau)d\tau}^{Q_{min}[E](t)} + c \overbrace{\int_{t}^{+\infty} E^2_{max}\left[E_t(0,*)\right](\tau)d\tau}^{F_{max}[E](t)}\,;$$

Generalized/Nonlocal R.H. Factorization Problem

$$\widehat{E_{max}}\,[E] = N_> \widehat{E},\, N_>(\omega) \in \mathcal{A}^+_n,\, 1/N_>(\omega) \in \mathcal{A}^-_n,\, N_>(\omega)N_>(-\omega) = n(\omega)$$

Designer Pulse for Optimal Energy Extraction: Semi-Infinite Slab $z \in [0, +\infty)$

$F_{min}[E](t)$

Designer Pulse for Optimal Energy Extraction: Semi-Infinite Slab $z \in [0, +\infty)$

$$F_{min}[E](t) := \max_{E_t^+} -\Delta_{[t,+\infty)} W[E_t^- + E_t^+]$$

Designer Pulse for Optimal Energy Extraction: Semi-Infinite Slab $z \in [0, +\infty)$

$$F_{min}[E](t) := \max_{E_t^+} -\Delta_{[t,+\infty)} W[E_t^- + E_t^+]$$

$$= W[E](t) - \min_{E_t^+} W[E_t^- + E_t^+](+\infty)$$

Designer Pulse for Optimal Energy Extraction: Semi-Infinite Slab $z \in [0, +\infty)$

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$W[E_t^- + E_t^+](+\infty)$

Designer Pulse for Optimal Energy Extraction: Semi-Infinite Slab $z \in [0, +\infty)$

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$$W[E_t^- + E_t^+](+\infty) = c \int_{-\infty}^{t} E_{min}^2 \left[E_t^-(0,*)\right](\tau) d\tau +$$

$$c \int_{t}^{+\infty} E_{min}^2 \left[\left(E_t^- + E_t^+\right)(0,*)\right](\tau) d\tau$$

Designer Pulse for Optimal Energy Extraction: Semi-Infinite Slab $z \in [0, +\infty)$

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$$c\int_{t}^{+\infty} E^2_{min}\left[\left(E_t^- + E_t^+\right)(0,*)\right](\tau)d\tau$$

$$\geq c\int_{-\infty}^{t} E^2_{min}\left[E_t^-(0,*)\right](\tau)d\tau$$

Designer Pulse for Optimal Energy Extraction: Semi-Infinite Slab $z \in [0, +\infty)$

$$F_{min}[E](t) := \max_{E_t^+} -\Delta_{[t,+\infty)} W[E_t^- + E_t^+]$$

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$$W[E_t^- + E_t^+](+\infty) = c\int_{-\infty}^{t} E_{min}^2\left[E_t^-(0,*)\right](\tau)d\tau +$$

$$c\int_{t}^{+\infty} E_{min}^2\left[\left(E_t^- + E_t^+\right)(0,*)\right](\tau)d\tau$$

$$\geq c\int_{-\infty}^{t} E_{min}^2\left[E_t^-(0,*)\right](\tau)d\tau$$

Designer Pulse for Optimal Energy Extraction: Semi-Infinite Slab $z \in [0, +\infty)$

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$$W[E_t^- + E_t^+](+\infty) = c \int_{-\infty}^{t} E^2_{min}\left[E_t^-(0,*)\right](\tau) d\tau +$$

$$0 = c \int_{t}^{+\infty} E^2_{min}\left[\left(E_t^- + E_t^+\right)(0,*)\right](\tau) d\tau$$

Designer Pulse for Optimal Energy Extraction: Semi-Infinite Slab $z \in [0, +\infty)$

$$F_{min}[E](t) := \max_{E_t^+} -\Delta_{[t,+\infty)} W[E_t^- + E_t^+]$$

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$$W[E_t^- + E_t^+](+\infty) = c\int_{-\infty}^{t} E_{min}^2\left[E_t^-(0,*)\right](\tau)d\tau +$$

$$0 = c\int_{t}^{+\infty} E_{min}^2\left[\left(E_t^- + E_t^+\right)(0,*)\right](\tau)d\tau$$

$$\Longleftrightarrow$$

Designer Pulse for Optimal Energy Extraction: Semi-Infinite Slab $z \in [0, +\infty)$

$$F_{min}[E](t) := \max_{E_t^+} -\Delta_{[t,+\infty)} W[E_t^- + E_t^+]$$

$$= W[E](t) - \min_{E_t^+} W[E_t^- + E_t^+](+\infty)$$

$$W[E_t^- + E_t^+](+\infty) = c \int_{-\infty}^{t} E_{min}^2 \left[E_t^-(0,*)\right](\tau) d\tau +$$

$$0 = c \int_{t}^{+\infty} E_{min}^2 \left[\left(E_t^- + E_t^+\right)(0,*)\right](\tau) d\tau$$

$$\Longleftrightarrow$$

$$0 = E_{min}\left[\left(E_t^- + E_t^+\right)(0,*)\right](\tau) \text{ for } \tau > t$$

Designer Pulse for Optimal Energy Extraction: Semi-Infinite Slab $z \in [0, +\infty)$

E_t^+

Designer Pulse for Optimal Energy Extraction: Semi-Infinite Slab $z \in [0, +\infty)$

E_t^+ :

Designer Pulse for Optimal Energy Extraction: Semi-Infinite Slab $z \in [0, +\infty)$

$E_t^+ \left[E_t^-(0, *) \right]$:

Designer Pulse for Optimal Energy Extraction: Semi-Infinite Slab $z \in [0, +\infty)$

$$E_t^+ \left[E_t^-(0,*)\right] \;:\; \mathcal{F}\left[E_{min}\right]\left[\left(E_t^- + E_t^+\right)(0,*)\right]$$

Designer Pulse for Optimal Energy Extraction: Semi-Infinite Slab $z \in [0, +\infty)$

$$E_t^+\left[E_t^-(0,*)\right] \ : \ \mathcal{F}\left[E_{min}\right]\left[\left(E_t^- + E_t^+\right)(0,*)\right] \in \mathcal{E}_t^-$$

Designer Pulse for Optimal Energy Extraction: Semi-Infinite Slab $z \in [0, +\infty)$

$$E_t^+ \left[E_t^-(0,*)\right] \;:\; \mathcal{F}\left[E_{min}\right]\left[\left(E_t^- + E_t^+\right)(0,*)\right] \in \mathcal{E}_t^- \qquad \Longleftrightarrow$$

Designer Pulse for Optimal Energy Extraction: Semi-Infinite Slab $z \in [0, +\infty)$

$$E_t^+ \left[E_t^-(0,*) \right] \;:\; \mathcal{F}\left[E_{min}\right]\left[\left(E_t^- + E_t^+ \right)(0,*) \right] \in \mathcal{E}_t^- \qquad \Longleftrightarrow$$
$$N_< \mathcal{F}\left[E_t^-(0,*) \right] \;+\; N_< \mathcal{F}\left[E_t^+(0,*) \right] \in \mathcal{E}_t^-$$

Designer Pulse for Optimal Energy Extraction: Semi-Infinite Slab $z \in [0, +\infty)$

$$E_t^+ \left[E_t^-(0,*)\right] \;:\; \mathcal{F}\left[E_{min}\right]\left[\left(E_t^- + E_t^+\right)(0,*)\right] \in \mathcal{E}_t^- \qquad \Longleftrightarrow$$

$$N_< \mathcal{F}\left[E_t^-(0,*)\right] \quad + \qquad N_< \mathcal{F}\left[E_t^+(0,*)\right] \in \mathcal{E}_t^- \qquad \Longrightarrow$$

Designer Pulse for Optimal Energy Extraction: Semi-Infinite Slab $z \in [0, +\infty)$

$$E_t^+ \left[E_t^-(0,*)\right] \ : \ \mathcal{F}[E_{min}]\left[\left(E_t^- + E_t^+\right)(0,*)\right] \in \mathcal{E}_t^- \iff$$

$$N_< \mathcal{F}\left[E_t^-(0,*)\right] + N_< \mathcal{F}\left[E_t^+(0,*)\right] \in \mathcal{E}_t^- \Longrightarrow$$

$$P_t^+ N_< \mathcal{F}\left[E_t^-(0,*)\right] + P_t^+ N_< \mathcal{F}\left[E_t^+(0,*)\right] = P_t^+ \mathcal{E}_t^- = 0$$

Designer Pulse for Optimal Energy Extraction: Semi-Infinite Slab $z \in [0, +\infty)$

$$E_t^+\left[E_t^-(0,*)\right] \;:\; \mathcal{F}\left[E_{min}\right]\left[\left(E_t^- + E_t^+\right)(0,*)\right] \in \mathcal{E}_t^- \qquad \Longleftrightarrow$$

$$N_<\mathcal{F}\left[E_t^-(0,*)\right] \quad + \quad N_<\mathcal{F}\left[E_t^+(0,*)\right] \in \mathcal{E}_t^- \qquad \Longrightarrow$$

$$P_t^+ N_<\mathcal{F}\left[E_t^-(0,*)\right] \quad + \quad P_t^+ N_<\mathcal{F}\left[E_t^+(0,*)\right] = P_t^+\mathcal{E}_t^- \quad = 0 \qquad \Longleftrightarrow$$

Designer Pulse for Optimal Energy Extraction: Semi-Infinite Slab $z \in [0, +\infty)$

$$\begin{aligned}
E_t^+\left[E_t^-(0,*)\right] \;:\; \mathcal{F}\left[E_{min}\right]\left[\left(E_t^- + E_t^+\right)(0,*)\right] &\in \mathcal{E}_t^- && \Longleftrightarrow \\
N_< \mathcal{F}\left[E_t^-(0,*)\right] \;+\; N_< \mathcal{F}\left[E_t^+(0,*)\right] &\in \mathcal{E}_t^- && \Longrightarrow \\
P_t^+ N_< \mathcal{F}\left[E_t^-(0,*)\right] \;+\; P_t^+ N_< \mathcal{F}\left[E_t^+(0,*)\right] &= P_t^+ \mathcal{E}_t^- = 0 && \Longleftrightarrow \\
P_t^+ N_< \mathcal{F}\left[E_t^-(0,*)\right] \;+\; N_< \mathcal{F}\left[E_t^+(0,*)\right] &= P_t^+ \mathcal{E}_t^- = 0 &&
\end{aligned}$$

Designer Pulse for Optimal Energy Extraction: Semi-Infinite Slab $z \in [0, +\infty)$

$$E_t^+ \left[E_t^-(0,*)\right] \; : \; \mathcal{F}\left[E_{min}\right]\left[\left(E_t^- + E_t^+\right)(0,*)\right] \in \mathcal{E}_t^- \quad \Longleftrightarrow$$

$$N_< \mathcal{F}\left[E_t^-(0,*)\right] + N_< \mathcal{F}\left[E_t^+(0,*)\right] \in \mathcal{E}_t^- \quad \Longrightarrow$$

$$P_t^+ N_< \mathcal{F}\left[E_t^-(0,*)\right] + P_t^+ N_< \mathcal{F}\left[E_t^+(0,*)\right] = P_t^+ \mathcal{E}_t^- = 0 \quad \Longleftrightarrow$$

$$P_t^+ N_< \mathcal{F}\left[E_t^-(0,*)\right] + N_< \mathcal{F}\left[E_t^+(0,*)\right] = P_t^+ \mathcal{E}_t^- = 0 \quad \Longleftrightarrow$$

Designer Pulse for Optimal Energy Extraction: Semi-Infinite Slab $z \in [0, +\infty)$

$$
\begin{aligned}
&E_t^+\left[E_t^-(0,*)\right] \;:\; \mathcal{F}\left[E_{min}\right]\left[\left(E_t^- + E_t^+\right)(0,*)\right] \in \mathcal{E}_t^- && \Longleftrightarrow \\
&N_<\mathcal{F}\left[E_t^-(0,*)\right] + N_<\mathcal{F}\left[E_t^+(0,*)\right] \in \mathcal{E}_t^- && \Longrightarrow \\
&P_t^+ N_<\mathcal{F}\left[E_t^-(0,*)\right] + P_t^+ N_<\mathcal{F}\left[E_t^+(0,*)\right] = P_t^+\mathcal{E}_t^- = 0 && \Longleftrightarrow \\
&P_t^+ N_<\mathcal{F}\left[E_t^-(0,*)\right] + N_<\mathcal{F}\left[E_t^+(0,*)\right] = P_t^+\mathcal{E}_t^- = 0 && \Longleftrightarrow
\end{aligned}
$$

$$
E_t^+(0,\tau) = -\mathcal{F}^{-1} \circ N_<^{-1} \circ P_t^+ \circ N_< \circ \mathcal{F}\left[E_t^-(0,*)\right](\tau);
$$

Designer Pulse for Optimal Energy Extraction: Semi-Infinite Slab $z \in [0, +\infty)$

$$
\begin{array}{rcll}
E_t^+\left[E_t^-(0,*)\right] \ : \ \mathcal{F}\left[E_{min}\right]\left[\left(E_t^- + E_t^+\right)(0,*)\right] \in \mathcal{E}_t^- & & & \Longleftrightarrow \\
N_<\mathcal{F}\left[E_t^-(0,*)\right] \ + \ N_<\mathcal{F}\left[E_t^+(0,*)\right] \in \mathcal{E}_t^- & & & \Longrightarrow \\
P_t^+N_<\mathcal{F}\left[E_t^-(0,*)\right] \ + \ P_t^+N_<\mathcal{F}\left[E_t^+(0,*)\right] = P_t^+\mathcal{E}_t^- & = 0 & & \Longleftrightarrow \\
P_t^+N_<\mathcal{F}\left[E_t^-(0,*)\right] \ + \ N_<\mathcal{F}\left[E_t^+(0,*)\right] = P_t^+\mathcal{E}_t^- & = 0 & & \Longleftrightarrow
\end{array}
$$

$$
\begin{aligned}
E_t^+(0,\tau) &= -\mathcal{F}^{-1} \circ N_<^{-1} \circ P_t^+ \circ N_< \circ \mathcal{F}\left[E_t^-(0,*)\right](\tau); \\
P_t^+ &= \mathcal{F} \circ \Theta_t^+ \circ \mathcal{F}^{-1}
\end{aligned}
$$

Designer Pulse for Optimal Energy Extraction: Semi-Infinite Slab $z \in [0, +\infty)$

Summary with Addendum:

$$F_{min}\,[E]\,(t) = \max_{E_t^+} -\Delta_{[t,+\infty)} W[E_t^- + E_t^+];$$

Designer Pulse for Optimal Energy Extraction: Semi-Infinite Slab $z \in [0, +\infty)$

Summary with Addendum:

$$F_{min}[E](t) = \max_{E_t^+} -\Delta_{[t,+\infty)} W[E_t^- + E_t^+];$$

$$E_t^+(0,\tau) = \qquad -\mathcal{F}^{-1} \circ N_<^{-1} \circ P_t^+ \circ N_< \circ \mathcal{F}\left[E_t^-(0,*)\right](\tau);$$

Designer Pulse for Optimal Energy Extraction: Semi-Infinite Slab $z \in [0, +\infty)$

Summary with Addendum:

$$F_{min}\,[E]\,(t) = \max_{E_t^+} -\Delta_{[t,+\infty)} W[E_t^- + E_t^+];$$

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Designer Pulse for Optimal Energy Extraction: Semi-Infinite Slab $z \in [0, +\infty)$

Summary with Addendum:

$$F_{min}[E](t) = \max_{E_t^+} -\Delta_{[t,+\infty)} W[E_t^- + E_t^+];$$

$$\begin{aligned} E_t^+(0,\tau) &= -\mathcal{F}^{-1} \circ N_<^{-1} \circ P_t^+ \circ N_< \circ \mathcal{F}\left[E_t^-(0,*)\right](\tau); \\ P_t^+ &= \mathcal{F} \circ \Theta_t^+ \circ \mathcal{F}^{-1}; \\ N_<(\omega) &= \exp \circ P_0^+ \circ \log n(\omega). \end{aligned}$$

Designer Pulse for Optimal Energy Injection: Semi-Infinite Slab $z \in [0, +\infty)$

$F_{max}\,[E]\,(t)$

Designer Pulse for Optimal Energy Injection: Semi-Infinite Slab $z \in [0, +\infty)$

$$F_{max}[E](t) := \min_{G_t^- \in \sigma(E_t^-)} W[G_t^-]$$

Designer Pulse for Optimal Energy Injection: Semi-Infinite Slab $z \in [0, +\infty)$

$$F_{max}\left[E\right](t) := \min_{G_t^- \in \sigma\left(E_t^-\right)} W[G_t^-] = c \int_t^{+\infty} E_{max}^2 \left[E_t^-(0,*)\right](\tau)$$

Designer Pulse for Optimal Energy Injection: Semi-Infinite Slab $z \in [0, +\infty)$

$$F_{max}[E](t) := \min_{G_t^- \in \sigma(E_t^-)} W[G_t^-] = c \int_t^{+\infty} E_{max}^2 \left[E_t^-(0,*)\right](\tau)$$

$W[G_t^-]$

Designer Pulse for Optimal Energy Injection: Semi-Infinite Slab $z \in [0, +\infty)$

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$$W[G_t^-] = c \int_{-\infty}^{t} E_{max}^2 \left[G_t^-(0,*)\right](\tau) d\tau + c \int_t^{+\infty} E_{max}^2 \left[G_t^-(0,*)\right](\tau) d\tau$$

Designer Pulse for Optimal Energy Injection: Semi-Infinite Slab $z \in [0, +\infty)$

$$F_{max}[E](t) := \min_{G_t^- \in \sigma(E_t^-)} W[G_t^-] = c \int_t^{+\infty} E^2_{max}\left[E_t^-(0,*)\right](\tau)$$

$$W[G_t^-] = c \int_{-\infty}^{t} E^2_{max}\left[G_t^-(0,*)\right](\tau) d\tau + c \int_t^{+\infty} E^2_{max}\left[E_t^-(0,*)\right](\tau) d\tau$$

Designer Pulse for Optimal Energy Injection: Semi-Infinite Slab $z \in [0, +\infty)$

$$F_{max}\left[E\right](t) := \min_{G_t^- \in \sigma\left(E_t^-\right)} W[G_t^-] = c \int_t^{+\infty} E_{max}^2 \left[E_t^-(0,*)\right](\tau)$$

$$\begin{aligned} W[G_t^-] = c \int_{-\infty}^{t} E_{max}^2 \left[G_t^-(0,*)\right](\tau) d\tau \, + \\ c \int_t^{+\infty} E_{max}^2 \left[E_t^-(0,*)\right](\tau) d\tau \\ \geq c \int_t^{+\infty} E_{max}^2 \left[E_t^-(0,*)\right](\tau) d\tau \end{aligned}$$

Designer Pulse for Optimal Energy Injection: Semi-Infinite Slab $z \in [0, +\infty)$

$$F_{max}\left[E\right](t) := \min_{G_t^- \in \sigma\left(E_t^-\right)} W[G_t^-] = c \int_t^{+\infty} E_{max}^2 \left[E_t^-(0,*)\right](\tau)$$

$$W[G_t^-] = c \int_{-\infty}^{t} E_{max}^2 \left[G_t^-(0,*)\right](\tau) d\tau +$$

$$c \int_t^{+\infty} E_{max}^2 \left[E_t^-(0,*)\right](\tau) d\tau$$

$$\geq c \int_t^{+\infty} E_{max}^2 \left[E_t^-(0,*)\right](\tau) d\tau$$

Designer Pulse for Optimal Energy Injection: Semi-Infinite Slab $z \in [0, +\infty)$

$$F_{max}[E](t) := \min_{G_t^- \in \sigma(E_t^-)} W[G_t^-] = c \int_t^{+\infty} E_{max}^2 \left[E_t^-(0,*)\right](\tau)$$

$$c \int_{-\infty}^{t} E_{max}^2 \left[G_t^-(0,*)\right](\tau) d\tau = 0$$

$$c \int_t^{+\infty} E_{max}^2 \left[G_t^-(0,*)\right](\tau) d\tau = c \int_t^{+\infty} E_{max}^2 \left[E_t^-(0,*)\right](\tau) d\tau$$

Designer Pulse for Optimal Energy Injection: Semi-Infinite Slab $z \in [0, +\infty)$

$$F_{max}\left[E\right](t) := \min_{G_t^- \in \sigma\left(E_t^-\right)} W[G_t^-] = c\int_t^{+\infty} E^2_{max}\left[E_t^-(0,*)\right](\tau)$$

$$c\int_{-\infty}^{t} E^2_{max}\left[G_t^-(0,*)\right](\tau)d\tau = 0$$

$$c\int_t^{+\infty} E^2_{max}\left[G_t^-(0,*)\right](\tau)d\tau = c\int_t^{+\infty} E^2_{max}\left[E_t^-(0,*)\right](\tau)d\tau$$

$$\Longleftrightarrow$$

Designer Pulse for Optimal Energy Injection: Semi-Infinite Slab $z \in [0, +\infty)$

$$F_{max}[E](t) := \min_{G_t^- \in \sigma(E_t^-)} W[G_t^-] = c \int_t^{+\infty} E_{max}^2 \left[E_t^-(0,*)\right](\tau)$$

$$c \int_{-\infty}^{t} E_{max}^2 \left[G_t^-(0,*)\right](\tau) d\tau = 0$$

$$c \int_t^{+\infty} E_{max}^2 \left[G_t^-(0,*)\right](\tau) d\tau = c \int_t^{+\infty} E_{max}^2 \left[E_t^-(0,*)\right](\tau) d\tau$$

$$\Longleftrightarrow$$

$$0 = E_{max}\left[G_t^-(0,*)\right](\tau) \text{ for } \tau < t$$

$$E_{max}\left[G_t^-(0,*)\right](\tau) = E_{max}\left[E_t^-(0,*)\right](\tau) \text{ for } \tau > t$$

Designer Pulse for Optimal Energy Injection: Semi-Infinite Slab $z \in [0, +\infty)$

$$G_t^- \left[E_t^-(0,*)\right] : N_> \mathcal{F}\left[G_t^-(0,*)\right] \in \mathcal{E}_t^+, \&$$
$$N_> \mathcal{F}\left[\left(G_t^- - E_t^-\right)(0,*)\right] \in \mathcal{E}_t^-;$$

Designer Pulse for Optimal Energy Injection: Semi-Infinite Slab $z \in [0, +\infty)$

$$
\begin{aligned}
G_t^- \left[E_t^-(0,*)\right] : \; & N_> \mathcal{F}\left[G_t^-(0,*)\right] \in \mathcal{E}_t^+, \& \\
& N_> \mathcal{F}\left[\left(G_t^- - E_t^-\right)(0,*)\right] \in \mathcal{E}_t^-; \\
& N_>(\omega) = \prod_{j=1}^{M} \frac{\left(\omega - \nu_j^*\right)\left(\omega + \nu_j\right)}{\left(\omega - \omega_j\right)\left(\omega + \omega_j^*\right)}
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Designer Pulse for Optimal Energy Injection: Semi-Infinite Slab $z \in [0, +\infty)$

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$$\mathcal{F}\left[G_t^-(0,*)\right](\omega) = \sum_{j=1}^{M} \frac{-iG_j}{\omega + \nu_j} + \frac{-iG_j^*}{\omega - \nu_j^*};$$

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$$\mathcal{F}\left[G_t^-(0,*)\right](\omega) = \mathcal{F}\left[E_t^-(0,*)\right](\omega), \qquad \omega = \omega_j, -\omega_j^*$$

Designer Pulse for Optimal Energy Injection: Semi-Infinite Slab $z \in [0, +\infty)$

Summary with Addenda:

$$F_{max}[E](t) := \min_{G_t^- \in \sigma(E_t^-)} W[G_t^-](+\infty)$$

Designer Pulse for Optimal Energy Injection: Semi-Infinite Slab $z \in [0, +\infty)$

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$$F_{max}\left[E\right](t) := \min_{G_t^- \in \sigma\left(E_t^-\right)} W[G_t^-](+\infty)$$

$$G_t^-(0,\tau) = \sqrt{2\pi}\left(\sum_{j=1}^{M} G_j \exp\left(i\nu_j\left(\tau - t\right)\right) + C.C.\right)\Theta_t^-(\tau)$$

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$n(\omega)$

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Designer Media for Optimal Energy Injection by a Specified Pulse: Semi-Infinite Slab $z \in [0, +\infty)$

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$$\frac{1}{2}\left(\sqrt{\varepsilon(\omega)} + \sqrt{\varepsilon(-\omega)}\right) = \overset{\checkmark}{n(\omega)}$$

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$$\frac{1}{2} \quad \sqrt{\varepsilon(\omega)} \qquad = \mathcal{F} \circ \Theta_0^+ \circ \mathcal{F}^{-1} \overset{\checkmark}{n(\omega)}$$

Complications for Optimal Energy Injection Designs: Finite Slab $z \in [0, L]$

$$N_{>}(\omega)N_{>}(-\omega) = \qquad n_{[0,L]}(\omega)$$

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\Longleftarrow &\quad \text{Global "Constitutive" Relation/Propagation:} \\
\widehat{E}''(z,\omega) = &\quad \frac{-\omega^2}{c^2}\left\{1 + \chi(\omega)\left[\Theta^+(z) - \Theta^+(z-L)\right]\right\}\widehat{E}\ (z,\omega);
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$$\widehat{E}''(z,\omega) = \frac{-\omega^2}{c^2}\left\{1+\chi(\omega)\left[\Theta^+(z) - \Theta^+(z-L)\right]\right\}\widehat{E}\ (z,\omega);$$

$\widehat{E}\ (0,\omega)$ prescribed.

Summary, Future Work

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  - Min Free Energy is maximum available for return to the field *after* some specified time

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- Future Prospects

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  - Min Free Energy is maximum available for return to the field *after* some specified time–Max Free Energy is minimum required to create current state *before* some specified time
  - Designing pulses and media for Max Extraction Energy involves only Kramers-Kronig-like calculus
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  - Designing media for Min Injection Energy is “hard” for the finite slab
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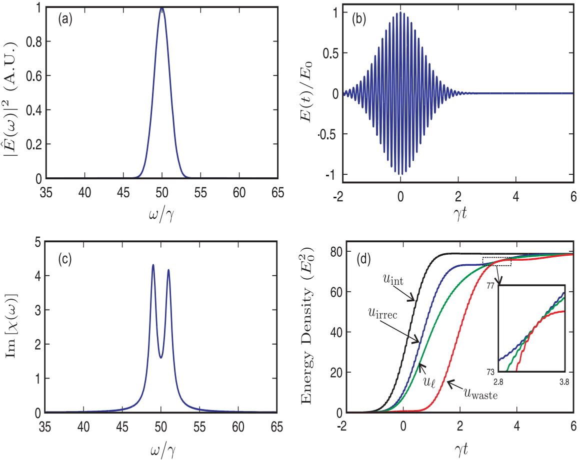
Broadband pulse encompassing two nearby absorption resonances



$$E(t) = E_0 e^{-t^2/T^2}\cos(\bar{\omega}t)$$

$$\chi(\omega) = \sum_{n=1}^{2} \frac{f_n \omega_{p_n}^2}{\omega_n^2 - i\gamma_n\omega - \omega^2}$$

$$\gamma_1 = \gamma_2 = \gamma\,;\ \omega_1 = 49\gamma$$

$$\omega_2 = 51\gamma\,;\ \bar{\omega} = 50\gamma$$

$$T = 1/\gamma.$$

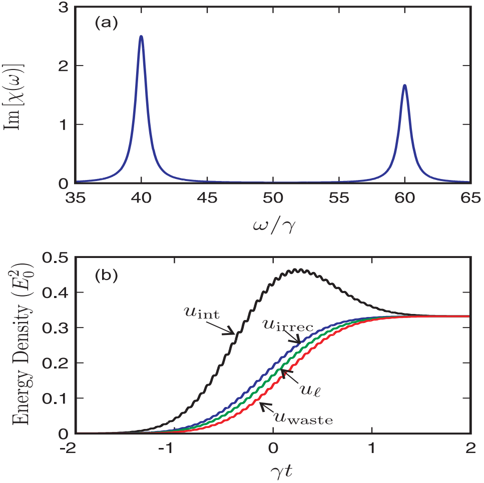
$$f_1\omega_{p_1}^2 = f_2\omega_{p_2}^2 = 200\gamma^2$$

$$u_{\mathrm{int}}(t) = W\,[E]\,(t)$$

$$u_{\mathrm{irrec}}(t) = Q_{\min}\,[E]\,(t)$$

$$u_{\mathrm{waste}}(t) = Q_{\max}\,[E]\,(t)$$

Two absorption resonances encompassing narrow-band pulse



$$E(t) = E_0 e^{-t^2/T^2} \cos(\bar{\omega} t)$$

$$\chi(\omega) = \sum_{n=1}^{2} \frac{f_n \omega_{p_n}^2}{\omega_n^2 - i\gamma_n \omega - \omega^2}$$

$$\gamma_1 = \gamma_2 = \gamma \,;\, \omega_1 = 40\gamma$$

$$\omega_2 = 60\gamma \,;\, \bar{\omega} = 50\gamma$$

$$T = 1/\gamma.$$

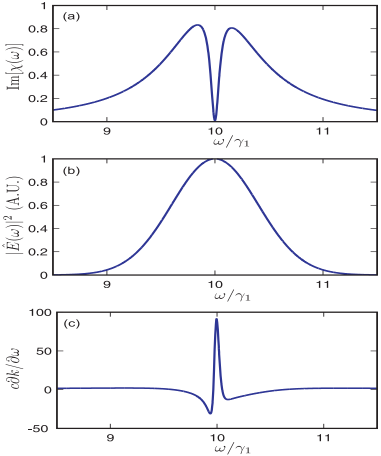
$$f_1 \omega_{p_1}^2 = f_2 \omega_{p_2}^2 = 100\gamma^2$$

$$u_{\rm int}(t) = W\,[E]\,(t)$$

$$u_{\rm irrec}(t) = Q_{\rm min}\,[E]\,(t)$$

$$u_{\rm waste}(t) = Q_{\rm max}\,[E]\,(t)$$

Broad-band pulse encompassing mixed passive/active resonances



$$E(t) = E_0 e^{-t^2/T^2} \cos(\bar{\omega} t)$$

$$\chi(\omega) = \sum_{n=1}^{2} \frac{f_n \omega_{p_n}^2}{\omega_n^2 - i\gamma_n \omega - \omega^2}$$

$$\gamma_2 = 0.1\gamma_1 \,;\, \omega_1 = \omega_2 = 10\gamma_1$$

$$\bar{\omega} = 10\gamma_1$$

$$T = 2.5/\gamma_1 .$$

$$f_1 \omega_{p_1}^2 = 10\gamma_1^2 \,;\, f_2 \omega_{p_2}^2 = -.99\gamma_1^2$$

$$u_{\mathrm{int}}(t) = W\,[E]\,(t)$$

$$u_{\mathrm{irrec}}(t) = Q_{\min}\,[E]\,(t)$$

$$u_{\mathrm{waste}}(t) = Q_{\max}\,[E]\,(t)$$

Broad-band pulse encompassing mixed passive/active resonances