Math 313 Midterm I KEY
Fall 2012
sections 008 and 011
Instructor: Scott Glasgow

Write your name very clearly on this exam booklet. In this booklet, write your mathematics clearly, legibly, in big fonts, and, most important, "have a point", i.e., make your work logically and even pedagogically acceptable. (Other human beings not already understanding 313 should be able to learn from your exam.) To avoid excessive erasing, first put your ideas together on scratch paper, then commit the logically acceptable fraction of your scratchings to this exam booklet. More is not necessarily better: say what you mean and mean what you say.

Honor Code: After I have learned of the contents of this exam by any means, I will not disclose to anyone any of these contents by any means until after the exam has closed. Also, I will not even speak of this exam to anyone else in *any* fashion, not even whether it was difficult or not. My signature below indicates I accept this obligation.

Signature:

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1. Put the following matrix in reduced row-echelon form via elementary row operations:

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 0 & 2 & 0 \\ 6 & -2 & 3 & 4 \end{bmatrix} \tag{1}$$

10pts

Solution

The row reduction might proceed as follows:

$$\begin{aligned} &\begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 0 & 2 & 0 \\ 6 & -2 & 3 & 4 \end{bmatrix}^{\substack{R2-2R1 \\ R3-6R1}} \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & 0 & -2 \\ 0 & -8 & -3 & -2 \end{bmatrix}^{\substack{R2/-2 \\ R3-4R2}} \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -3 & 6 \end{bmatrix}^{R3/-3} \\ &\sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix}^{R1-R3} \sim \begin{bmatrix} 1 & 1 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix}^{R1-R2} \sim \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix} \end{aligned} \tag{2}$$

However the row reduction proceeds, the row echelon form is unique—the last matrix indicated in (2) is ***the*** answer.

2. Give an expression for the general solution to the system of equations

$$\begin{aligned} w + x + y + z &= 0 \\ 2w \quad + 2y \quad &= 0 \\ 6w - 2x + 3y + 4z &= 0. \end{aligned} \tag{3}$$

For the final 5 points of this problem, insert your expression for the solution into (3) to make sure that it works. That is, check your answer by plugging it back into (3).

5+5pts

Solution

The augmented matrix, less the augmentation, is the matrix in (1). Thus, according to the last matrix in (2), (3) is equivalent to

$$\begin{aligned} w \qquad\qquad + 2z &= 0 \\ x \qquad + z &= 0 \\ y - 2z &= 0, \end{aligned} \tag{4}$$

whose general solution is clearly

$$\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2t \\ -t \\ 2t \\ t \end{bmatrix} = t \begin{bmatrix} -2 \\ -1 \\ 2 \\ 1 \end{bmatrix} =: t\mathbf{x}_0. \tag{5}$$

Since (3) is homogeneous, in order to check that (5) is an expression of at least some of its solutions, it is enough to there plug in

$$\mathbf{x}_0 := \begin{bmatrix} -2 \\ -1 \\ 2 \\ 1 \end{bmatrix} =: \begin{bmatrix} w_0 \\ x_0 \\ y_0 \\ z_0 \end{bmatrix}. \tag{6}$$

We find

$$\begin{aligned} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 0 & 2 & 0 \\ 6 & -2 & 3 & 4 \end{bmatrix} \mathbf{x}_0 &= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 0 & 2 & 0 \\ 6 & -2 & 3 & 4 \end{bmatrix} \begin{bmatrix} w_0 \\ x_0 \\ y_0 \\ z_0 \end{bmatrix} \\ &= \begin{bmatrix} w_0 + x_0 + y_0 + z_0 \\ 2w_0 + 2y_0 \\ 6w_0 - 2x_0 + 3y_0 + 4z_0 \end{bmatrix} \\ &= \begin{bmatrix} -2 - 1 + 2 + 1 \\ 2(-2) + 2(2) \\ 6(-2) - 2(-1) + 3(2) + 4(1) \end{bmatrix} \\ &= \begin{bmatrix} -3 + 3 \\ -4 + 4 \\ -12 + 2 + 6 + 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -10 + 10 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \end{aligned} \tag{7}$$

It works. This does not of course check that there aren't more solutions.

3. Give an expression for the general solution to the system of equations

$$\begin{aligned} x+y+z&=1 \\ 2x+2z&=0 \\ 6x-2y+3z&=4. \end{aligned} \tag{8}$$

For the final 5 points of this problem, insert your expression for the solution into (8) to make sure that it works. That is, check your answer by plugging it back into(8).

5+5pts

Solution

The augmented matrix for (8) is the matrix in (1), hence by the last matrix in (2) the unique solution of (8) is

$$\mathbf{x}=\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}. \tag{9}$$

We check that

$$\begin{aligned} x+y+z&=2+1+(-2)=3-2=1 \\ 2x+2z&=2(2)+2(-2)=4-4=0 \\ 6x-2y+3z&=6(2)-2(1)+3(-2)=12-2-6=10-6=4. \end{aligned} \tag{10}$$

It works.

4. Suppose column vectors $\mathbf{x}_1$ and $\mathbf{x}_2$ solve the linear system

$$A\mathbf{x}=\mathbf{b}. \tag{11}$$

That is, suppose both that

$$A\mathbf{x}_1=\mathbf{b} \tag{12}$$

and that

$$A\mathbf{x}_2=\mathbf{b}. \tag{13}$$

Now define a new column vector $\mathbf{x}_3$ from the equation

$$\mathbf{x}_3 := t\mathbf{x}_2+(1-t)\mathbf{x}_1=\mathbf{x}_1+t(\mathbf{x}_2-\mathbf{x}_1). \tag{14}$$

Here t is any real number. Show that 'interpolation' $\mathbf{x}_3$ also solves (11). (By doing so, you will have shown that when a linear system has two distinct solutions, it actually has infinitely many, namely all interpolations between the two original solutions.)

10 pts

Solution

By matrix algebra and then using (12) and (13) we readily find that

$$\begin{aligned} A\mathbf{x}_3 &= A\left(t\mathbf{x}_2 + (1-t)\,\mathbf{x}_1\right) = tA\mathbf{x}_2 + (1-t)\,A\mathbf{x}_1 = t\mathbf{b} + (1-t)\,\mathbf{b} \\ &= \left(t + (1-t)\right)\mathbf{b} = 1\mathbf{b} = \mathbf{b}. \end{aligned} \tag{15}$$

5. Assuming A and B are invertible matrices of the same size, prove that

$$(AB)^{-1} = B^{-1}A^{-1}. \tag{16}$$

10 points

Solution

$B^{-1}A^{-1}$ is the inverse of AB if and only if

$$(AB)\left(B^{-1}A^{-1}\right) = \left(B^{-1}A^{-1}\right)(AB) = I, \tag{17}$$

to whit we first note that, by the associative property of matrix multiplication,

$$\begin{gathered} (AB)\left(B^{-1}A^{-1}\right) = A\left(\left(BB^{-1}\right)A^{-1}\right) \\ \text{and} \\ \left(B^{-1}A^{-1}\right)(AB) = B^{-1}\left(\left(A^{-1}A\right)B\right). \end{gathered} \tag{18}$$

Then, by the definition of the inverses, in particular that an inverse is both a right and a left inverse, we have, respectively, that

$$\begin{gathered} (AB)\left(B^{-1}A^{-1}\right) = A\left(\left(BB^{-1}\right)A^{-1}\right) = A\left(IA^{-1}\right) \\ \text{and} \\ \left(B^{-1}A^{-1}\right)(AB) = B^{-1}\left(\left(A^{-1}A\right)B\right) = B^{-1}(IB). \end{gathered} \tag{19}$$

Using now the fact that the identity matrix is in fact the "multiplicative identity" we get

$$\left(AB\right)\left(B^{-1}A^{-1}\right)=A\left(IA^{-1}\right)=AA^{-1}$$
$$\text{and} \tag{20}$$
$$\left(B^{-1}A^{-1}\right)\left(AB\right)=B^{-1}\left(IB\right)=B^{-1}B.$$

Finally we use again the definition of the inverses. In particular, using that an inverse is both a right and a left inverse, we have, respectively, that

$$\left(AB\right)\left(B^{-1}A^{-1}\right)=AA^{-1}=I$$
$$\text{and} \tag{21}$$
$$\left(B^{-1}A^{-1}\right)\left(AB\right)=B^{-1}B=I,$$

which is the required (17).

6. Use Gaussian elimination, noting along the way the various relevant row operations and their relationship to the (evolving calculation of) the determinant, to compute

$$\det\begin{bmatrix} 9h & 2p & 4a & y & P \\ 9h & 2p & 5a & y & P \\ 11h & 3p & 6a & y & P \\ 15h & 4p & 8a & y & 2P \\ 8h & 2p & 4a & y & P \end{bmatrix} \tag{22}$$

   by reducing it to an upper triangular matrix (the determinant of which being the product of its diagonal entries).

   Be sure to note along the way the various relevant row operations to your Gaussian elimination and their accurate relationship to the (evolving calculation of) the determinant. (Note also a lower case p and an upper case P, whose values are potentially distinct. That is, these are independent variables.)

10 points

<u>Solution</u>

We indicate the various Gaussian elimination row operations in the upper right-hand corner of the matrix (whose determinant is being calculated), making sure to use the correct relationship of such to the evolving determinant calculation:

$$\det\begin{bmatrix} 9h & 2p & 4a & y & P \\ 9h & 2p & 5a & y & P \\ 11h & 3p & 6a & y & P \\ 15h & 4p & 8a & y & 2P \\ 8h & 2p & 4a & y & P \end{bmatrix} \overset{\substack{R1-R5\\R2-R5\\R3-R5\\R4-2R5}}{=} \det\begin{bmatrix} h & 0 & 0 & 0 & 0 \\ h & 0 & a & 0 & 0 \\ 3h & p & 2a & 0 & 0 \\ -h & 0 & 0 & -y & 0 \\ 8h & 2p & 4a & y & P \end{bmatrix}$$

$$\overset{\substack{R2-R1\\R3-3R1\\R4+R1\\R5-8R1}}{=} \det\begin{bmatrix} h & 0 & 0 & 0 & 0 \\ 0 & 0 & a & 0 & 0 \\ 0 & p & 2a & 0 & 0 \\ 0 & 0 & 0 & -y & 0 \\ 0 & 2p & 4a & y & P \end{bmatrix} \overset{R5-2R3}{=} \det\begin{bmatrix} h & 0 & 0 & 0 & 0 \\ 0 & 0 & a & 0 & 0 \\ 0 & p & 2a & 0 & 0 \\ 0 & 0 & 0 & -y & 0 \\ 0 & 0 & 0 & y & P \end{bmatrix}$$

$$\overset{R5+R4}{=} \det\begin{bmatrix} h & 0 & 0 & 0 & 0 \\ 0 & 0 & a & 0 & 0 \\ 0 & p & 2a & 0 & 0 \\ 0 & 0 & 0 & -y & 0 \\ 0 & 0 & 0 & 0 & P \end{bmatrix} \overset{R2\leftrightarrow R3}{=} -\det\begin{bmatrix} h & 0 & 0 & 0 & 0 \\ 0 & p & 2a & 0 & 0 \\ 0 & 0 & a & 0 & 0 \\ 0 & 0 & 0 & -y & 0 \\ 0 & 0 & 0 & 0 & P \end{bmatrix} \tag{23}$$

$$= -hpa(-y)P = hapPy.$$

7. Recall the (permutation) definition of a determinant of a (square) matrix can be equally unambiguously communicated as a ‘wedge product’ of the rows of the matrix, and that such makes proving a determinant’s properties under row operations easier (than doing so by permutations). For example, using that this product is associative and linear, we can easily prove that if B is the same as square matrix A, except that one of B’s rows is a scalar multiple k of the corresponding row of A, then

$$\det B = k \det A. \tag{24}$$

Here’ how we easily prove this using wedge products: Write

$$\begin{aligned} \det B &:= \mathbf{b}_1 \wedge \ldots \wedge \mathbf{b}_j \wedge \ldots \wedge \mathbf{b}_n = \mathbf{a}_1 \wedge \ldots \wedge \left(k\mathbf{a}_j\right) \wedge \ldots \wedge \mathbf{a}_n \\ &= k\left(\mathbf{a}_1 \wedge \ldots \wedge \mathbf{a}_j \wedge \ldots \wedge \mathbf{a}_n\right) =: k \det A. \end{aligned} \tag{25}$$

It’s as simple as that! This used the associativity property in the sense that it was not important in (25) to use parentheses to indicate which adjacent wedge product was to be done first, then second, etc. It also used linearity ‘weakly’, the full/strong version of which being that, by definition of a wedge product,

$$(c\mathbf{a}+k\mathbf{b})\wedge\mathbf{c}=c\ \mathbf{a}\wedge\mathbf{c}+k\ \mathbf{b}\wedge\mathbf{c}. \tag{26}$$

To prove that swapping any two rows gives the opposite sign in a determinant is a little trickier in general, but depends rather directly/simply on the only 'groovy'/unique property of a wedge product, namely that swapping arguments gives the opposite sign: by definition of wedge product we have

$$\boldsymbol{\beta}\wedge\boldsymbol{\alpha}=-\boldsymbol{\alpha}\wedge\boldsymbol{\beta}. \tag{27}$$

Note an immediate fact arising from (27) is that

$$\boldsymbol{\alpha}\wedge\boldsymbol{\alpha}=0. \tag{28}$$

Why? Because from replacing $\boldsymbol{\beta}$ in (27) with $\boldsymbol{\alpha}$ we immediately have

$$\boldsymbol{\alpha}\wedge\boldsymbol{\alpha}=-\boldsymbol{\alpha}\wedge\boldsymbol{\alpha} \tag{29}$$

and then by adding $\boldsymbol{\alpha}\wedge\boldsymbol{\alpha}$ to both sides of (29) we get

$$2\ \boldsymbol{\alpha}\wedge\boldsymbol{\alpha}=0 \tag{30}$$

and then (28) follows after division of both sides of (30) by two. The reason we care about property (28) of a wedge product is that, when using the wedge product definition of determinant, it is what dictates that when two rows of a determinant are the same, the determinant is zero: skipping the proof steps, the final result is that

$$\det\begin{bmatrix}\mathbf{r}_1\\ \vdots\\ \boldsymbol{\alpha}\\ \vdots\\ \boldsymbol{\alpha}\\ \vdots\\ \mathbf{r}_n\end{bmatrix}:=\mathbf{r}_1\wedge\ldots\wedge\boldsymbol{\alpha}\wedge\ldots\wedge\boldsymbol{\alpha}\wedge\ldots\wedge\mathbf{r}_n=0. \tag{31}$$

Your mission, should you choose to accept it, is to use (31) and linearity (26) to (easily!) prove that when you add a multiple κ of one row to another row of a (square) matrix you do absolutely nothing to the determinant of that matrix. Mark, get set, go!

5 points

<u>Solution</u>

Here we imagine a multiple of an 'earlier' row added to a 'later' row, but the proof the other way round is identical:

$$\begin{aligned}\det B &:= \mathbf{b}_1 \wedge \ldots \wedge \mathbf{b}_j \wedge \ldots \wedge \mathbf{b}_k \wedge \ldots \wedge \mathbf{b}_n = \mathbf{a}_1 \wedge \ldots \wedge \mathbf{a}_j \wedge \ldots \wedge \left(\mathbf{a}_k + \kappa \mathbf{a}_j\right) \wedge \ldots \wedge \mathbf{a}_n \\ &= \mathbf{a}_1 \wedge \ldots \wedge \mathbf{a}_j \wedge \ldots \wedge \mathbf{a}_k \wedge \ldots \wedge \mathbf{a}_n + \kappa\left(\mathbf{a}_1 \wedge \ldots \wedge \mathbf{a}_j \wedge \ldots \wedge \mathbf{a}_j \wedge \ldots \wedge \mathbf{a}_n\right) \\ &= \mathbf{a}_1 \wedge \ldots \wedge \mathbf{a}_j \wedge \ldots \wedge \mathbf{a}_k \wedge \ldots \wedge \mathbf{a}_n + \kappa \cdot 0 \\ &= \mathbf{a}_1 \wedge \ldots \wedge \mathbf{a}_j \wedge \ldots \wedge \mathbf{a}_k \wedge \ldots \wedge \mathbf{a}_n =: \det A. \end{aligned} \tag{32}$$

8. Let A be a square matrix, and consider the linear system

$$A\mathbf{x} = \mathbf{b}. \tag{33}$$

The usual theorem is that if A is invertible—which means 'left and right invertible'—then (33) has exactly one solution, namely

$$\mathbf{x} = A^{-1}\mathbf{b}. \tag{34}$$

a) Imagine a universe in which A has a left inverse B, but is not guaranteed to have a right one. So we have

$$BA = I \tag{35}$$

but we are not at all guaranteed that there is a matrix C (be it B or any other matrix) such that $AC = I$. Now prove one of the following, whichever one is possible (it is not possible to prove both with only (35)):

i) Prove that (33) has at least one solution $\mathbf{x}$, namely $\mathbf{x} = B\mathbf{b}$.
ii) Prove that (33) has at most one solution $\mathbf{x}$, namely $\mathbf{x} = B\mathbf{b}$.

b) Imagine a universe in which A has a right inverse C, but is not guaranteed to have a left one. So we have

$$AC = I \tag{36}$$

but we are not at all guaranteed that there is a matrix B (be it C or any other matrix) such that $BA = I$. Now prove one of the following, whichever one is possible (it is not possible to prove both with only (36)):

i) Prove that (33) has at least one solution $\mathbf{x}$, namely $\mathbf{x} = C\mathbf{b}$.
ii) Prove that (33) hat at most one solution $\mathbf{x}$, namely $\mathbf{x} = C\mathbf{b}$.

10 points

Solution

a) **If** the system has a solution $\mathbf{x}$, then, for any such $\mathbf{x}$,we may write

$$A\mathbf{x} = \mathbf{b} \tag{37}$$

(without implicitly lying), and then, by left application of B to (37), as well as by the associative property of matrix multiplication, obtain that

$$B\mathbf{b} = B(A\mathbf{x}) = (BA)\mathbf{x} = I\mathbf{x} = \mathbf{x}, \tag{38}$$

i.e., (37)/(33) implies

$$\mathbf{x} = B\mathbf{b}. \tag{39}$$

In (38) we also used that B is a left inverse of A, i.e., we used (35), as well as the fact that the so-called identity matrix I is in fact a "multiplicative identity". Here then we have just showed that **if** (37)/(33) has a solution, it's got to be $\mathbf{x} = B\mathbf{b}$. Thus we have showed that (37)/(33) has at most one solution, namely $\mathbf{x} = B\mathbf{b}$. This is the second option ii) above. We have not at all proved that (39) actually solves (37). The latter is impossible to show without knowing B is a right inverse of A.

b) With right inverse C satisfying (36) we easily show that (37)/(33) has at least one solution, namely $\mathbf{x} = C\mathbf{b}$: we simply note that with $\mathbf{x} = C\mathbf{b}$ we certainly get

$$A\mathbf{x} = A(C\mathbf{b}) = (AC)\mathbf{b} = I\mathbf{b} = \mathbf{b}. \tag{40}$$

This is the first option i) above. This does not at all prove that there aren't other solutions. The latter is impossible to show without knowing that C is a left inverse of A.

9. Prove that the product AB of two square matrices (of the same size) cannot be invertible if B is not invertible. That is, prove that

$$B \text{ is not invertible} \Rightarrow AB \text{ is not invertible.} \tag{41}$$

Hint: Consider the relationship between solutions $\mathbf{x}$ of the equation $B\mathbf{x} = \mathbf{0}$ and solutions $\mathbf{x}$ of the equation $(AB)\mathbf{x} = \mathbf{0}$, and use equivalent statements.

10 points

Solution

By equivalent statements and associativity of matrix multiplication we have

$$B \text{ is not invertible} \overset{\text{eq. stat.s}}{\Leftrightarrow} \exists\, \mathbf{x} \neq \mathbf{0} \text{ s.t. } B\mathbf{x} = \mathbf{0}$$
$$\Rightarrow \exists\, \mathbf{x} \neq \mathbf{0} \text{ s.t} (AB)\mathbf{x} \overset{\text{assoc.}}{=} A(B\mathbf{x}) = A(\mathbf{0}) = \mathbf{0} \tag{42}$$
$$\overset{\text{eq. stat.s}}{\Leftrightarrow} AB \text{ is not invertible.}$$

10. Assume that both the matrix B and the matrix C are inverses of the matrix A. Show that B and C are just two aliases for the same matrix, i.e. show that in fact $B = C$.

10 points

Solution

The descriptions of B and C demand that

$$AB = BA = I = AC = CA. \tag{43}$$

Using the associative property of matrix multiplication in two different ways on the product BAC we get

$$\begin{gathered} BAC = B(AC) = BI = B \\ \text{and} \\ BAC = (BA)C = IC = C, \end{gathered} \tag{44}$$

so that indeed

$$\begin{gathered} B = BAC = C \\ \Rightarrow \\ B = C \end{gathered} \tag{45}$$

as claimed. Note that in (44) we also used that a) C is a right inverse of A, b) B is a left inverse of A, and that c) the identity matrix acts as both a right and left multiplicative identity. If we wanted to use that C is a left inverse of A and that B is a right inverse in our proof, we could have done the following similar (but still different) computations:

$$\begin{gathered} CAB = (CA)B = IB = B \\ \text{and} \\ CAB = C(AB) = CI = C. \end{gathered} \tag{46}$$

11. Prove that $(AB)^T = B^T A^T$ for any two (not necessarily square) matrices A and B (for which the product AB makes sense).

10pts

Solution

By definition of matrix product and transpose we have both

$$\begin{gathered}\left((AB)^T\right)_{ij} := (AB)_{ji} := \sum_k A_{jk} B_{ki}, \\ \text{and} \\ \left(B^T A^T\right)_{ij} := \sum_k \left(B^T\right)_{ik} \left(A^T\right)_{kj} := \sum_k B_{ki} A_{jk} = \sum_k A_{jk} B_{ki},\end{gathered} \tag{47}$$

which gives

$$\left((AB)^T\right)_{ij} = \left(B^T A^T\right)_{ij} \tag{48}$$

for any and all i and j, and which then gives $(AB)^T = B^T A^T$ by definition of matrix equality.

12. Find the inverse of the matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \tag{49}$$

by row reducing $\left[A \middle| I\right]$ to $\left[I \middle| A^{-1}\right]$. Assume the parameters $a, b, c,$ and d do not take on any special values, nor have a special relationship among them—that is row reduce naively, without worrying about any divisions by hidden zeros. Simplify your answer completely. (Related: kA^{-1} should be exceptionally simple if scalar k is chosen to be $ad - bc$.)

10 points

Solution

The naïve row reduction mentioned might proceed as follows:

$$\left[A|I\right] = \left[\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right]^{aR2-cR1} \sim \left[\begin{array}{cc|cc} a & b & 1 & 0 \\ 0 & ad-bc & -c & a \end{array}\right]^{(ad-bc)R1-bR2}$$

$$\sim \left[\begin{array}{cc|cc} a(ad-bc) & 0 & ad & -ab \\ 0 & ad-bc & -c & a \end{array}\right]^{R1/a} \sim \left[\begin{array}{cc|cc} ad-bc & 0 & d & -b \\ 0 & ad-bc & -c & a \end{array}\right]^{\substack{R1/(ad-bc) \\ R2/(ad-bc)}} \tag{50}$$

$$\sim \left[\begin{array}{cc|c} 1 & 0 & \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \\ 0 & 1 & \end{array}\right]^{R1/a} = \left[I|A^{-1}\right].$$

In any case, the generally applicable expression

$$\left[A|I\right] \sim \left[I|A^{-1}\right] = \left[\begin{array}{cc|c} 1 & 0 & \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \\ 0 & 1 & \end{array}\right] \tag{51}$$

is in fact unique.

13. Suppose the system of equations

$$A\mathbf{x} = \mathbf{b} \tag{52}$$

has one and only one solution $\mathbf{x} \in \mathbb{R}^n$ for each and every $\mathbf{b} \in \mathbb{R}^n$. (Evidently A is an $n \times n$ matrix.) So, tell me whether the following statement is true or is false: “It is possible that the matrix A allows there to be $\mathbf{x} \neq \mathbf{0}$ such that $A\mathbf{x} = \mathbf{0}$”, i.e. “It is possible that the homogeneous version of equation (52) has a nontrivial solution”. ***Prove your assertion.***

5 points

<u>Solution</u>

Without recourse to any “fancy theorems”, we have the following: $A\mathbf{x} = \mathbf{0}$ for $\mathbf{x} = \mathbf{0}$ certainly, and the hypotheses say that this is the only one. So the statement is false.

14. Suppose the system of equations

$$A\mathbf{x} = \mathbf{b} \tag{53}$$

has a solution $\mathbf{x} \in \mathbb{R}^n$ for each and every $\mathbf{b} \in \mathbb{R}^n$. (Evidently A is an $n \times n$ matrix.) So, tell me whether the following statement is true or is false: “It is possible that the matrix A allows there to be $\mathbf{x} \neq \mathbf{0}$ such that $A\mathbf{x} = \mathbf{0}$”, i.e. “It is possible that the homogeneous version of equation (53) has a nontrivial solution”. ***Prove your assertion.***

10 points

Solution

With recourse to our "Equivalent Statements", we have that $A\mathbf{x}=\mathbf{0}$ has only the trivial solution $\mathbf{x}=\mathbf{0}$ if and only if (53) has a solution $\mathbf{x}\in\mathbb{R}^n$ for every $\mathbf{b}\in\mathbb{R}^n$. So the statement is false.

Extra Pedagogy: Recall that the proof of this equivalence goes something like this: If (53) is consistent for every choice of $\mathbf{b}\in\mathbb{R}^n$, then we can solve systems $A\mathbf{x}_i=\mathbf{b}_i$, $i=1,\ldots,n$, with the $\mathbf{b}_i$'s being the relevant columns of the identity matrix I. Then the matrix $C=\left[\mathbf{x}_1\middle|\ldots\middle|\mathbf{x}_i\middle|\ldots\middle|\mathbf{x}_n\right]$ certainly turns out to be a "right inverse" of A, which, by theorem 1.6.3, will also be "***the*** inverse" of A, so that $A\mathbf{x}=\mathbf{0}$ has only the solution $\mathbf{x}=I\mathbf{x}=\left(A^{-1}A\right)\mathbf{x}=A^{-1}\left(A\mathbf{x}\right)=A^{-1}\mathbf{0}=\mathbf{0}$. [Note that theorem 1.6.3 could itself be proved there by showing that this right inverse is itself invertible—consider the system $C\mathbf{x}=\mathbf{0}$, which then has only the solution $\mathbf{x}=I\mathbf{x}=\left(AC\right)\mathbf{x}=A\left(C\mathbf{x}\right)=A\mathbf{0}=\mathbf{0}$, which, according to equivalent statements, gives C invertible—and then using that, under this circumstance, $AC=I\Rightarrow A=C^{-1}\Rightarrow CA=CC^{-1}=I$, which says (among other things) that C is also A's inverse.] This is enough of the equivalence to deduce that the statement is false.

To get the other part of this equivalence, and to illuminate how some of the other equivalent statements just used are indeed equivalent, recall the following: if $A\mathbf{x}=\mathbf{0}$ has only the trivial solution $\mathbf{x}=\mathbf{0}$, then row reduction of $\left[A\middle|\mathbf{0}\right]$ must give $\left[I\middle|\mathbf{0}\right]$ (which presumes that row reduction does not change the solution space, which we have never really proved but seems clear) , i.e. row reduction of A must give I (ignoring the last columns in the above augmented matrices), which shows (with theorem 1.5.1) that the product of A with a finite number of elementary matrices is I, which then shows that A can be expressed as a product of (the inverses of the) elementary matrices, and, so, is itself invertible, giving (finally!) that $A\left(A^{-1}\mathbf{b}\right)=\left(AA^{-1}\right)\mathbf{b}=I\mathbf{b}=\mathbf{b}$ for every $\mathbf{b}\in\mathbb{R}^n$, so that (53) has at least the solution $\mathbf{x}=A^{-1}\mathbf{b}$ for every $\mathbf{b}$. Aside from row reduction preserving the solution space of a system of equations, the other "big idea" that may be buried in here is that fact that elementary matrices, or, more to the point, elementary row operations, are "truly" invertible, i.e. that the left or right inverses of such are in fact also right and left inverses. This last statement formed in terms of elementary row operations is the following: not only is it the case that for every elementary row operation there is another one that will "undo" it "afterwards", but that same "afterwards inverse" done "before" the given elementary row operation will itself be undone by the given elementary row operation. Of course this distinction of "before" and "after" is at the heart of what we mean by "right" and "left" inverses. Whew!

Note that at one very basic level, the truth of all of our equivalent statements comes down to row operations, specifically that they don't alter the solution space of a system of equations, and (very much related) that they are "before/after"= "left\right" invertible.

Perhaps these last two (but certainly related) claims should be thoroughly investigated by the serious student.

15. Suppose the system of equations

$$A\mathbf{x} = \mathbf{b} \tag{54}$$

has at most 1 solution $\mathbf{x} \in \mathbb{R}^n$ for each and every $\mathbf{b} \in \mathbb{R}^n$. (Evidently A is an $n \times n$ matrix.) So, tell me whether the following statement is true or is false: "It is possible that the matrix A allows there to be a $\mathbf{b} \in \mathbb{R}^n$ such that (54) is inconsistent." Said differently, this statement is "It is possible that the matrix A is so special that there is a $\mathbf{b} \in \mathbb{R}^n$ for which (54) has 0 solutions." Note that "having 0 solutions" does not disagree with the statement that (54) "has at most 1 solution". ***Prove your assertion.***

10 points

<u>Solution</u>

If (54) has at most one solution for each $\mathbf{b} \in \mathbb{R}^n$, then, choosing $\mathbf{b} = \mathbf{0} \in \mathbb{R}^n$, we see that the system $A\mathbf{x} = \mathbf{0}$ has only the solution $\mathbf{x} = \mathbf{0}$: $\mathbf{x} = \mathbf{0}$ certainly solves the equation ($A\mathbf{0} = \mathbf{0}$) and we've just said that there can't be any other solutions. But, by equivalent statements again, (55) is consistent for every $\mathbf{b} \in \mathbb{R}^n$. Thus the statement is false again!

16. Suppose the system of equations

$$A\mathbf{x} = \mathbf{b} \tag{55}$$

has no solution $\mathbf{x} \in \mathbb{R}^n$ for some particular $\mathbf{b} \in \mathbb{R}^n$. (Here A is an $n \times n$ matrix.) Tell me whether the following statement is true or is false: "There is another right-hand-side $\mathbf{b} \in \mathbb{R}^n$ such that (55) has infinitely many solutions." ***Prove your assertion***.

10 points

<u>Solution</u>

By (the contrapositive/negation of our stated) equivalent statements the supposition gives us that, for example, $A\mathbf{x} = \mathbf{0}$ has more than just one solution. (Here we have chosen the "other $\mathbf{b}$" to be $\mathbf{0}$. Mind you $\mathbf{0}$ really is *another* $\mathbf{b}$ since for $\mathbf{b} = \mathbf{0}$ (55) actually has a solution, namely $\mathbf{x} = \mathbf{0}$, contrary to the supposition.) But since the possibilities for the number of solutions of linear systems are only 0, 1, or ∞, there must be an infinite number of solutions for $\mathbf{b} = \mathbf{0}$, so that the statement is true.