Math 313 Midterm II KEY
Fall 2012
sections 008 and 011
Instructor: Scott Glasgow

Write your name very clearly on this exam. In this booklet, write your mathematics clearly, legibly, in big fonts, and, most important, "have a point", i.e. make your work logically and even pedagogically acceptable. (Other human beings not already understanding 313 should be able to learn from your exam.) To avoid excessive erasing, first put your ideas together on scratch paper, then commit the logically acceptable fraction of your scratchings to this exam booklet. More is not necessarily better: say what you mean and mean what you say.

Honor Code: After I have learned of the contents of this exam by any means, I will not disclose to anyone any of these contents by any means until after the exam has closed. My signature below indicates I accept this obligation.

Signature:

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1) Let A be a square matrix, and consider the linear system

$$A\mathbf{x} = \mathbf{b}. \tag{1}$$

The usual theorem is that if A is invertible—which means 'left and right invertible'—then (1) has exactly one solution, namely

$$\mathbf{x} = A^{-1}\mathbf{b}. \tag{2}$$

a) Imagine a universe in which A has a left inverse B, but is not guaranteed to have a right one. So we have

$$BA = I \tag{3}$$

but we are not at all guaranteed that there is a matrix C (be it B or any other matrix) such that $AC = I$. Now prove one of the following, whichever one is possible: (It is not possible to prove both with only (3).)

i) Prove that (1) has at least one solution $\mathbf{x}$, namely $\mathbf{x} = B\mathbf{b}$.
ii) Prove that (1) has at most one solution $\mathbf{x}$, namely $\mathbf{x} = B\mathbf{b}$.

b) Imagine a universe in which A has a right inverse C, but is not guaranteed to have a left one. So we have

$$AC = I \tag{4}$$

but we are not at all guaranteed that there is a matrix B (be it C or any other matrix) such that $BA = I$. Now prove one of the following, whichever one is possible: (It is not possible to prove both with only (4).)

i) Prove that (1) has at least one solution $\mathbf{x}$, namely $\mathbf{x} = C\mathbf{b}$.
ii) Prove that (1) hat at most one solution $\mathbf{x}$, namely $\mathbf{x} = C\mathbf{b}$.

10 points

Solution

a) **If** the system has a solution $\mathbf{x}$, then, for any such $\mathbf{x}$, we may write

$$A\mathbf{x} = \mathbf{b} \tag{5}$$

(without implicitly lying), and then, by left application of B to (5), as well as by the associative property of matrix multiplication, obtain that

$$B\mathbf{b} = B\left(A\mathbf{x}\right) = \left(BA\right)\mathbf{x} = I\mathbf{x} = \mathbf{x}, \tag{6}$$

i.e., (5)/(1) implies

$$\mathbf{x} = B\mathbf{b}. \tag{7}$$

In (6) we also used that B is a left inverse of A, i.e., we used (3), as well as the fact that the so-called identity matrix I is in fact a "multiplicative identity". Here then we have just showed that **if** (5)/(1) has a solution, it's got to be $\mathbf{x} = B\mathbf{b}$. Thus we have showed that (5)/(1) has at most one solution, namely $\mathbf{x} = B\mathbf{b}$. This is the second option ii) above. We have not at all proved that (7) actually solves (5). The latter is impossible to show without knowing B is a right inverse of A.

b) With right inverse C satisfying (4) we easily show that (5)/(1) has at least one solution, namely $\mathbf{x} = C\mathbf{b}$: we simply note that with $\mathbf{x} = C\mathbf{b}$ we certainly get

$$A\mathbf{x} = A\left(C\mathbf{b}\right) = \left(AC\right)\mathbf{b} = I\mathbf{b} = \mathbf{b}. \tag{8}$$

This is the first option i) above. This does not at all prove that there aren't other solutions. The latter is impossible to show without knowing that C is a left inverse of A.

2) Show that

$$S' := \left\{\left(1,2,3\right),\left(4,5,6\right),\left(7,8,10\right)\right\} \tag{9}$$

is a basis for $\mathbb{R}^3$. Cite any labor-saving theorems used. Also, after **sufficient** explanation of any possible relevance, you may use **either** that the reduced row-echelon form of

$$A := \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 10 \end{bmatrix} \tag{10}$$

is the identity

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{11}$$

or that the reduced row-echelon form of

$$B := \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 10 \end{bmatrix} \tag{12}$$

is the identity (as in (11)). Using the wrong one of either A or B will result in lost points—as will not explaining accurately why the reduced row-echelon form of whichever matrix (A or B) is even relevant in the first place, including not citing the relevant theorem accurately.

<u>10pts</u>

<u>Solution</u>

$\mathbb{R}^3$ is three dimensional:

$$S := \{(1,0,0),(0,1,0),(0,0,1)\} =: \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\} \tag{13}$$

is a basis. S' is a set of three vectors in this three dimensional space and, so, theorem, to show that S' is a basis for $\mathbb{R}^3$ it is enough to show that S' spans $\mathbb{R}^3$, or show that S' is linearly independent. The latter is arguably easier:

By definition, S' is linearly independent iff the only solution $(k_1, k_2, k_3) \in \mathbb{R}^3$ of the equation

$$k_1(1,2,3) + k_2(4,5,6) + k_3(7,8,10) = (0,0,0) \tag{14}$$

is $(k_1, k_2, k_3) = (0,0,0)$. Now

$$\begin{aligned} k_1(1,2,3) + k_2(4,5,6) + k_3(7,8,10) &= (k_1, 2k_1, 3k_1) + (4k_2, 5k_2, 6k_2) + (7k_3, 8k_3, 10k_3) \\ &= (k_1 + 4k_2 + 7k_3, 2k_1 + 5k_2 + 8k_3, 3k_1 + 6k_2 + 10k_3) \end{aligned} \tag{15}$$

so (14) is equivalent to

$$(k_1 + 4k_2 + 7k_3, 2k_1 + 5k_2 + 8k_3, 3k_1 + 6k_2 + 10k_3) = (0,0,0), \tag{16}$$

which is itself equivalent to the system of equations

$$\begin{aligned} k_1 + 4k_2 + 7k_3 &= 0, \\ 2k_1 + 5k_2 + 8k_3 &= 0, \\ 3k_1 + 6k_2 + 10k_3 &= 0, \end{aligned} \tag{17}$$

which system can be rewritten as

$$A\mathbf{x} := \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 10 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \mathbf{0}. \tag{18}$$

Here we defined

$$A := \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 10 \end{bmatrix}, \qquad \mathbf{x} := \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} \tag{19}$$

in order to make a connection via familiar notation: From (18) we see that S' is linearly independent hence a basis iff this " $A\mathbf{x} = \mathbf{0}$ " system (linear, homogeneous system of as many equations as unknowns) has only the trivial solution

$$\mathbf{x} := \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \tag{20}$$

And by equivalent statements the latter holds iff the reduced-row echelon form of A is

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{21}$$

which it is by (10)/(11) (not (12)/(11)).

3) Suppose that $S := \{(1,1,1,1), (1,1,1,-1), (1,1,-1,-1)\}$ is a basis for a subspace W of $\mathbb{R}^4$. Show that

$$S' := \{(6,6,0,-4), (15,15,3,-7), (25,25,5,-11))\} \tag{22}$$

is also a basis for W. You may use that

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 10 \end{bmatrix} = \begin{bmatrix} 6 & 15 & 25 \\ 6 & 15 & 25 \\ 0 & 3 & 5 \\ -4 & -7 & -11 \end{bmatrix}, \tag{23}$$

i.e., you may use that

$$\begin{bmatrix}1&1&1\\1&1&1\\1&1&-1\\1&-1&-1\end{bmatrix}\begin{bmatrix}1\\2\\3\end{bmatrix}=\begin{bmatrix}6\\6\\0\\-4\end{bmatrix},\begin{bmatrix}1&1&1\\1&1&1\\1&1&-1\\1&-1&-1\end{bmatrix}\begin{bmatrix}4\\5\\6\end{bmatrix}=\begin{bmatrix}15\\15\\3\\-7\end{bmatrix},\begin{bmatrix}1&1&1\\1&1&1\\1&1&-1\\1&-1&-1\end{bmatrix}\begin{bmatrix}7\\8\\10\end{bmatrix}=\begin{bmatrix}24\\24\\6\\-10\end{bmatrix}. \tag{24}$$

You may also use **either** that the reduced row echelon form of

$$\begin{bmatrix}6&15&25\\6&15&25\\0&3&5\\-4&-7&-11\end{bmatrix} \tag{25}$$

is

$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\\0&0&0\end{bmatrix} \tag{26}$$

or that the reduced row echelon form of

$$\begin{bmatrix}6&6&0&-4\\15&15&3&-7\\25&25&5&-11\end{bmatrix} \tag{27}$$

is

$$\begin{bmatrix}1&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}. \tag{28}$$

As in problem 1), one of these facts ((25)/(26) or (27)/(28)) is relevant, the other irrelevant.

<u>10pts</u>

<u>Solution</u>

W is 3 dimensional and S' is a set of three vectors, so provided $S' \subset W$, S' is a basis for W if it is linearly independent or spans W, the former easier to prove. Now (24) shows that each of the vectors in S' is the indicated linear combination of vectors in S, hence gives $S' \subset W$. Following parts of problem 1) we find S' is linearly independent iff the system

$$A\mathbf{x} := \begin{bmatrix} 6 & 15 & 24 \\ 6 & 15 & 24 \\ 0 & 3 & 6 \\ -4 & -7 & -10 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{29}$$

has only the trivial solution

$$\begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \tag{30}$$

Here we defined

$$A := \begin{bmatrix} 6 & 15 & 24 \\ 6 & 15 & 24 \\ 0 & 3 & 6 \\ -4 & -7 & -10 \end{bmatrix}, \qquad \mathbf{x} := \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix}. \tag{31}$$

(29) has only the trivial solution (30) iff there are no free variables, i.e., iff every column of the reduced row echelon form of A has a leading one. From (25)/(26)—***<u>NOT</u>*** (27)/(28)) —we see that that is the case.

4) Suppose $S := \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis for a vector space V . Show that

$$S' := \{\mathbf{v}_1', \mathbf{v}_2', \mathbf{v}_3'\} := \{\mathbf{v}_1 + 2\mathbf{v}_2 + 3\mathbf{v}_3, 4\mathbf{v}_1 + 5\mathbf{v}_2 + 6\mathbf{v}_3, 7\mathbf{v}_1 + 8\mathbf{v}_2 + 10\mathbf{v}_3\} \tag{32}$$

is also a basis for V .

<u>10pts</u>

<u>Solution</u>

V is 3 dimensional and S' is a set of three vectors in V (using $S := \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ spans V), so S' is a basis provided it is linearly independent or spans V , the former easier to prove: Consider whether the equation

$$k_1' \cdot \mathbf{v}_1' + k_2' \cdot \mathbf{v}_2' + k_3' \cdot \mathbf{v}_3' = \mathbf{z} \tag{33}$$

has only the trivial solution $\left(k_1', k_2', k_3'\right) = (0,0,0)$ (which solution it certainly has). From (32) we see (33) is equivalent to both

$$k_1' \cdot \left(\mathbf{v}_1 + 2\mathbf{v}_2 + 3\mathbf{v}_3\right) + k_2' \cdot \left(4\mathbf{v}_1 + 5\mathbf{v}_2 + 6\mathbf{v}_3\right) + k_3' \cdot \left(7\mathbf{v}_1 + 8\mathbf{v}_2 + 10\mathbf{v}_3\right) = \mathbf{z}, \tag{34}$$

and then, via vector space algebra,

$$\left(k_1' + 4k_2' + 7k_3'\right) \cdot \mathbf{v}_1 + \left(2k_1' + 5k_2' + 8k_3'\right) \cdot \mathbf{v}_2 + \left(3k_1' + 6k_2' + 10k_3'\right) \cdot \mathbf{v}_3 = \mathbf{z}, \tag{35}$$

which is

$$k_1 \cdot \mathbf{v}_1 + k_2 \cdot \mathbf{v}_2 + k_3 \cdot \mathbf{v}_3 = \mathbf{z} \tag{36}$$

where

$$k_1 := k_1' + 4k_2' + 7k_3', \qquad k_2 := 2k_1' + 5k_2' + 8k_3', \qquad k_3 := 3k_1' + 6k_2' + 10k_3'. \tag{37}$$

But since $S := \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis, (36) implies (and is in fact equivalent to) $\left(k_1, k_2, k_3\right) = (0,0,0)$, which, with (37) is the system of equations

$$\begin{aligned} k_1' + 4k_2' + 7k_3' &= 0, \\ 2k_1' + 5k_2' + 8k_3' &= 0, \\ 3k_1' + 6k_2' + 10k_3' &= 0, \end{aligned} \tag{38}$$

which can be rewritten as in (18) (with primes on the k ’s), and which, by (10)/(11), does indeed have exactly the trivial solution.

5) Let

$$A = \begin{bmatrix} 6 & 6 & 0 & -4 \\ 15 & 15 & 3 & -7 \\ 25 & 25 & 5 & -11 \end{bmatrix}. \tag{39}$$

Find a basis for the row space of A **among the rows of** A**.** (You will docked points if you find a different type of basis.) Now find a basis for the column space of A **among the columns of** A. (You will docked points if you find a different type of basis.) Finally find a basis for the nullspace of A.

<u>10 points</u>

<u>Solution</u>

By (27)/(28), the matrix A above is row equivalent to

$$R = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{40}$$

and, so, theorem, the row space of A is that of R, a basis for the latter (clearly) being all R's rows since none is zero: The theorem is that nonzero rows—the rows with leading ones—of a reduced row echelon matrix form a basis for its row space. (The theorem also states that the columns of reduced row echelon matrix with leading ones form a basis for its column space. The proof of either part of this theorem is a trivial analysis involving only a discussion of the location of 1's and 0's.) So

$$S_r := \{(1,1,0,0),(0,0,1,0),(0,0,0,1)\} \tag{41}$$

is such a basis. Of course, if any of the rows of A were dependent, the dimension of the row space couldn't be 3—by the plus minus theorem we could remove the 'dependent' row and get the same span. So it must be all A's rows are independent too and we have that A's 'original' rows also form a basis. Thus this gives ***the*** answer to the first question:

$$row(A) = Span\{(6,6,0,-4),(15,15,3,-7),(25,25,5,-11)\} =: Span\ S, \tag{42}$$

where S is a basis (consisting of all the rows of A). But this answer was obtained in a roundabout way. A more direct way would be to note that

$$A^T = \begin{bmatrix} 6 & 15 & 25 \\ 6 & 15 & 25 \\ 0 & 3 & 5 \\ -4 & -7 & -11 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} =: R_T, \tag{43}$$

where $\sim$ means "row equivalent to". According to the theorem mentioned (in parentheses) below (40), we see that R_T's 'first three' columns—all of its columns—form a basis for the column space of R_T. This is actually obvious without a theorem. Less obvious is that, theorem, the 'first three' columns of the row-equivalent matrix A^T—all of A^T's columns—must therefore form a basis for the column space of A^T. Of course these are just the 'first three' rows of A—i.e., all of A's rows—and we get (42) 'directly'.

To answer the second question in the way demanded—and rather directly this time—we note that a basis for the column space of R in (40) is it's first, third and fourth columns—either by the 'theorem in parentheses' again, or by the fact that that's just obvious. Therefore, theorem, the first, third and fourth columns of A form a basis for the column space of A:

$$col(A) = Span\left\{\begin{bmatrix}6\\15\\25\end{bmatrix},\begin{bmatrix}0\\3\\5\end{bmatrix},\begin{bmatrix}-4\\-7\\-11\end{bmatrix}\right\} =: Span\ S', \tag{44}$$

where S' is a basis (consisting of columns of A). [Of course we could have taken the second, third and fourth columns just as well. ;-)]

Let's look at the third question now. By theorem, the null space of A is that of R's, and, so we get

$$\begin{aligned} nul(A) &= \left\{\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}\middle| \begin{matrix}x_1+x_2=0\\ x_3=0\\ x_4=0\end{matrix}\ ,\ x_2\in\mathbb{R}\right\} = \left\{\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}\middle| \begin{matrix}x_1=-x_2\\ x_3=0\\ x_4=0\end{matrix}\ ,\ x_2\in\mathbb{R}\right\} \\ &= \left\{\begin{bmatrix}-x_2\\x_2\\0\\0\end{bmatrix}\middle| x_2\in\mathbb{R}\right\} = \left\{x_2\begin{bmatrix}-1\\1\\0\\0\end{bmatrix}\middle| x_2\in\mathbb{R}\right\} =: Span\left\{\begin{bmatrix}-1\\1\\0\\0\end{bmatrix}\right\} = Span\left\{\begin{bmatrix}1\\-1\\0\\0\end{bmatrix}\right\}. \end{aligned} \tag{45}$$

In (45) we clearly have a basis—a single vector forms a linearly independent set unless it's the zero vector. This is easy to see in the concrete setting of $\mathbb{R}^n$ (in this case $\mathbb{R}^4$), but also follows from the general theorem that, in a vector space,

$$k\cdot\mathbf{u}=\mathbf{z},\quad \mathbf{u}\neq\mathbf{z}\quad\Rightarrow\quad k=0, \tag{46}$$

where $\mathbf{z}$ is the additive identity in the space. (Of course we effectively have a theorem that, in any event, this procedure is going to produce a linearly independent spanning set—a basis.)

6) Consider the following set S of vectors in $\mathbb{R}^4$. Explain why S is linearly dependent without doing any calculations. Next, give a basis for the subspace $W=\text{Span}\ S$ and use this basis for W to express one of the vectors in S as a linear

combination of ***others*** in S. (No fair saying a vector is 1 times itself.) Finally, what is the dimension of $W = \text{Span } S$?

$$S = \left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 4 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 2 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix} \right\} \tag{47}$$

You may use the fact that

$$A := \begin{bmatrix} 1 & 1 & 2 & 1 & 1 \\ 1 & 0 & 3 & 3 & 1 \\ 2 & 2 & 4 & 2 & 2 \\ 0 & 1 & -1 & -2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 & 3 & 0 \\ 0 & 1 & -1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} =: B, \tag{48}$$

where the tilde ($\sim$) indicates "row equivalent to".

<u>**10pts**</u>

<u>**Solution**</u>

S is 5 vectors from $\mathbb{R}^4$, so since $5 > 4 = \dim \mathbb{R}^4$, theorem, S is dependent. Next, since B is in reduced row echelon form, its pivot columns (clearly) define a basis for its column space, all other columns linear combinations then of these special columns. These pivot columns are its first, second and fifth. And since, theorem, row reduction does not alter the linear relationships among columns of a matrix, the associated columns of A are a basis for the column space of A: the first, second and fifth columns of A give a basis for the column space of A. So since these columns are the first, second and fifth elements of S, it must be that the first, second and fifth elements of S are a basis for $W = \text{Span } S$: a basis for $W = \text{Span } S$ is the set

$$S' = \left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix} \right\}. \tag{49}$$

Here we see then that $\dim W = \dim \text{Span } S = \dim \text{Span } S' = |S'| = 3$, which answers the last question.

With the theory just presented, we have that since

$$[\mathbf{b}_1 \quad \mathbf{b}_2 \quad \mathbf{b}_3 \quad \mathbf{b}_4 \quad \mathbf{b}_5] := \begin{bmatrix} 1 & 0 & 3 & 3 & 0 \\ 0 & 1 & -1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{50}$$

gives

$$\mathbf{b}_3 = (3)\mathbf{b}_1 + (-1)\mathbf{b}_2 = 3\mathbf{b}_1 - \mathbf{b}_2, \quad \mathbf{b}_4 = (3)\mathbf{b}_1 + (-2)\mathbf{b}_2 = 3\mathbf{b}_1 - 2\mathbf{b}_2, \tag{51}$$

it must be that

$$\begin{bmatrix} 2 \\ 3 \\ 4 \\ -1 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 3 \\ 2 \\ -2 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix} - 2\begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \end{bmatrix}, \tag{52}$$

either one of which two statements answering then the second question.

7) For the given set V of objects, together with the indicated notions of addition and scalar multiplication, determine whether each of the ten vector space axioms holds: V is real pairs (x, y), where

$$\begin{aligned} (x, y) + (x', y') &:= (x + x', y + y'), \\ k \cdot (x, y) &:= (2kx - 2ky, kx - ky), \end{aligned} \tag{53}$$

10pts

Solution

Axioms 1) through 5) should hold since they reference only vector addition and since $V = \mathbb{R}^2$ with only scalar multiplication altered.

6) $k \cdot (x, y) \in V = \mathbb{R}^2$ when $(x, y) \in V = \mathbb{R}^2$ and $k \in \mathbb{R}$ since both $2kx - 2ky$ and $kx - ky$ are clearly real numbers then.

7) First of all note that

$$k \cdot (x, y) := (2k(x - y), k(x - y)) \tag{54}$$

and that we then have

$$\begin{aligned}
k\cdot\left((x,y)+(x',y')\right) &:= k\cdot(x+x',y+y') \\
&:= (2k\left(x+x'-\left(y+y'\right)\right),k\left(x+x'-\left(y+y'\right)\right)) \\
&= (2k\left(x-y\right)+2k\left(x'-y'\right),k\left(x-y\right)+k\left(x'-y'\right)) \\
&=: (2k\left(x-y\right),k\left(x-y\right))+(2k\left(x'-y'\right),k\left(x'-y'\right)) \\
&=: k\cdot(x,y)+k\cdot(x',y').
\end{aligned} \tag{55}$$

So this axiom holds.

For axiom 8) we have

$$\begin{aligned}
\left(k+m\right)\cdot(x,y) &:= (2\left(k+m\right)\left(x-y\right),\left(k+m\right)\left(x-y\right)) \\
&= (\left(2k+2m\right)\left(x-y\right),\left(k+m\right)\left(x-y\right)) \\
&= (2k\left(x-y\right)+2m\left(x-y\right),k\left(x-y\right)+m\left(x-y\right)) \\
&=: (2k\left(x-y\right),k\left(x-y\right))+(2m\left(x-y\right),m\left(x-y\right)) \\
&=: k\cdot(x,y)+m\cdot(x,y).
\end{aligned} \tag{56}$$

So this axiom holds too.

For axiom 9) we have

$$\begin{aligned}
k\cdot\left(m\cdot(x,y)\right) &:= k\cdot(2m\left(x-y\right),m\left(x-y\right)) \\
&:= (2k\left(2m\left(x-y\right)-m\left(x-y\right)\right),k\left(2m\left(x-y\right)-m\left(x-y\right)\right)) \\
&= (2k\left(2m-m\right)\left(x-y\right),k\left(2m-m\right)\left(x-y\right)) \\
&= (2km\left(x-y\right),km\left(x-y\right)) \\
&= (2\left(km\right)\left(x-y\right),\left(km\right)\left(x-y\right)) =: \left(km\right)\cdot(x,y).
\end{aligned} \tag{57}$$

So this axiom holds also.

Finally then note that with regard to axiom 10) we have

$$1\cdot(x,y) := (2\left(x-y\right),1\left(x-y\right)) = (2\left(x-y\right),x-y) \not\equiv (x,y). \tag{58}$$

In particular note that

$$1\cdot(1,1) = (2\left(1-1\right),1-1) = (0,0) \neq (1,1). \tag{59}$$

Thus all axioms hold except the last.

8) Show that if $S=\left\{\mathbf{v}_1,\mathbf{v}_2\right\}$ be a basis for a vector space V. Show that $W=\left\{\mathbf{w}_1,\mathbf{w}_2,\mathbf{w}_3\right\}$, which is a set of three vectors in V, is linearly dependent.

10pts

Solution

We consider whether

$$c_1 \cdot \mathbf{w}_1 + c_2 \cdot \mathbf{w}_2 + c_3 \cdot \mathbf{w}_3 = \mathbf{z} \Rightarrow c_1 = c_2 = c_3 = 0. \tag{60}$$

Apparently we will find (60) ***doesn't*** hold, i.e., we will find there exists $(c_1, c_2, c_3) \neq (0,0,0)$ such that the left hand side of (60) holds. Here's how we can do that.

Well, since $S = \{\mathbf{v}_1, \mathbf{v}_2\}$ is a basis for V, it spans V, which means there are scalars $k_{11}, k_{21}; k_{12}, k_{22}; k_{13}, k_{23}$ such that

$$\mathbf{w}_1 = k_{11} \cdot \mathbf{v}_1 + k_{21} \cdot \mathbf{v}_2, \quad \mathbf{w}_2 = k_{12} \cdot \mathbf{v}_1 + k_{22} \cdot \mathbf{v}_2, \quad \mathbf{w}_3 = k_{13} \cdot \mathbf{v}_1 + k_{23} \cdot \mathbf{v}_2, \tag{61}$$

and the left-hand-side of (60) is equivalent to

$$\begin{aligned} \mathbf{z} &= c_1 \cdot \mathbf{w}_1 + c_2 \cdot \mathbf{w}_2 + c_3 \cdot \mathbf{w}_3 \\ &= c_1 \cdot (k_{11} \cdot \mathbf{v}_1 + k_{21} \cdot \mathbf{v}_2) + c_2 \cdot (k_{12} \cdot \mathbf{v}_1 + k_{22} \cdot \mathbf{v}_2) + c_3 \cdot (k_{13} \cdot \mathbf{v}_1 + k_{23} \cdot \mathbf{v}_2) \\ &= (k_{11}c_1 + k_{12}c_2 + k_{13}c_3)\mathbf{v}_1 + (k_{21}c_1 + k_{22}c_2 + k_{23}c_3)\mathbf{v}_2, \end{aligned} \tag{62}$$

i.e., it's equivalent to

$$(k_{11}c_1 + k_{12}c_2 + k_{13}c_3)\mathbf{v}_1 + (k_{21}c_1 + k_{22}c_2 + k_{23}c_3)\mathbf{v}_2 = \mathbf{z}. \tag{63}$$

Now, by theorem, this equation—which is equivalent to the left-hand-side of (60)—certainly holds if

$$\begin{aligned} k_{11}c_1 + k_{12}c_2 + k_{13}c_3 &= 0, \\ k_{21}c_1 + k_{22}c_2 + k_{23}c_3 &= 0. \end{aligned} \tag{64}$$

With the k's given and the c's being sought, (64) is a system of two linear homogeneous equations in 3 unknowns, hence has a solution $(c_1, c_2, c_3) \neq (0,0,0)$, (60) doesn't hold, and $W = \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ is linearly dependent as advertized.

9) Let W be the set of all simultaneous solutions $\mathbf{x}\in\mathbb{R}^n$ of two linear, homogeneous systems of equations:

$$W=\left\{\mathbf{x}\in\mathbb{R}^n \middle| A\mathbf{x}=\mathbf{0}_m\in\mathbb{R}^m \quad\text{and}\quad B\mathbf{x}=\mathbf{0}_l\in\mathbb{R}^l\right\}. \tag{65}$$

(Evidently A is an $m\times n$ matrix, and B is a $l\times n$ matrix.) It turns out that W is a subspace of $\mathbb{R}^n$. Prove this. Use any ***relevant*** theorems learned in this class, except the one you developed in a HW problem regarding intersections of vector spaces. (Don't just say, "We proved that the intersection of two subspaces of a vector space is a subspace.") In avoiding use of that HW problem result, you will effectively prove the very general thing anew, but think your're just doing something specific, which will make the problem seem easier.

<u>10pts</u>

<u>Solution</u>

Since $W\subset\mathbb{R}^n$, by theorem it will be a subspace provided it's not empty, and provided it is closed under linear combos. Well, W is nonempty because $\mathbf{x}=\mathbf{0}_n\in\mathbb{R}^n$ solves both $A\mathbf{x}=\mathbf{0}_m\in\mathbb{R}^m$ and $B\mathbf{x}=\mathbf{0}_l\in\mathbb{R}^l$. Now let's see if W is closed under linear combination: Let $\mathbf{x},\mathbf{y}\in W$ and let $\alpha,\beta\in\mathbb{R}$ all be arbitrary. W will be a subspace if $\alpha\mathbf{x}+\beta\mathbf{y}\in W$. Well, since $\mathbf{x},\mathbf{y}\in W$, all of the following equations hold true:

$$\begin{aligned} A\mathbf{x}&=\mathbf{0}_m \quad, &\quad B\mathbf{x}&=\mathbf{0}_l, \\ A\mathbf{y}&=\mathbf{0}_m \quad, &\quad B\mathbf{y}&=\mathbf{0}_l. \end{aligned} \tag{66}$$

So then

$$\begin{aligned} A(\alpha\mathbf{x}+\beta\mathbf{y})&=\alpha A\mathbf{x}+\beta A\mathbf{y}=\alpha\mathbf{0}_m+\beta\mathbf{0}_m=\mathbf{0}_m, \\ B(\alpha\mathbf{x}+\beta\mathbf{y})&=\alpha B\mathbf{x}+\beta B\mathbf{y}=\alpha\mathbf{0}_l+\beta\mathbf{0}_l=\mathbf{0}_l, \end{aligned} \tag{67}$$

and, so, by definition, $\alpha\mathbf{x}+\beta\mathbf{y}\in W$ and we do indeed get W is a subspace of $\mathbb{R}^n$. (In (67) we first used matrix algebra, then that any linear combo of the same type of zero vector is that type of zero vector.)

10) Let A be an $m\times n$ matrix. Write it as

$$A=\begin{bmatrix}\mathbf{r}_1^T\\ \vdots \\ \mathbf{r}_m^T\end{bmatrix} \tag{68}$$

where $\mathbf{r}_1^T, \ldots, \mathbf{r}_m^T$ are m row vectors of "width" n, i.e., $\{\mathbf{r}_1^T, \ldots, \mathbf{r}_m^T\}$ is a set of m ***row*** vectors, each in $\mathbb{R}^n$. It will also be useful to realize then that $\{\mathbf{r}_1, \ldots, \mathbf{r}_m\}$ is a set of m ***column*** vectors, each one in $\mathbb{R}^n$. Recall that

$$row(A) := Span\{\mathbf{r}_1^T, \ldots, \mathbf{r}_m^T\} := \{c_1\mathbf{r}_1^T + \ldots + c_m\mathbf{r}_m^T \mid c_1, \ldots, c_m \in \mathbb{R}\}. \tag{69}$$

Of course we can ignore the distinction between row and column vectors and just write

$$row(A) := Span\{\mathbf{r}_1, \ldots, \mathbf{r}_m\} := \{c_1\mathbf{r}_1 + \ldots + c_m\mathbf{r}_m \mid c_1, \ldots, c_m \in \mathbb{R}\}. \tag{70}$$

Also recall that

$$nul(A) := \{\mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} = \mathbf{0}_m \in \mathbb{R}^m\}. \tag{71}$$

But now note that we can write

$$\begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} = \mathbf{0}_m = A\mathbf{x} = \begin{bmatrix} \mathbf{r}_1^T \\ \vdots \\ \mathbf{r}_m^T \end{bmatrix} \mathbf{x} = \begin{bmatrix} \mathbf{r}_1^T\mathbf{x} \\ \vdots \\ \mathbf{r}_m^T\mathbf{x} \end{bmatrix} = \begin{bmatrix} \mathbf{r}_1 \cdot \mathbf{x} \\ \vdots \\ \mathbf{r}_m \cdot \mathbf{x} \end{bmatrix}, \tag{72}$$

whence

$$\mathbf{x} \in nul(A) \Leftrightarrow \mathbf{r}_1 \cdot \mathbf{x} = \ldots = \mathbf{r}_m \cdot \mathbf{x} = 0, \tag{73}$$

(provided $\mathbf{x} \in \mathbb{R}^n$ so the dot product makes sense). Of course we can write all this as

$$nul(A) := \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{r}_1 \cdot \mathbf{x} = \ldots = \mathbf{r}_m \cdot \mathbf{x} = 0\} \tag{74}$$

where A is given by (68).

(73) says "$\mathbf{x}$ is in the nullspace of A iff $\mathbf{x}$ is orthogonal to each of A's rows". Now the "orthogonal complement" of $row(A)$, denoted $(row(A))^{\perp}$, is "the set of all vectors in $\mathbb{R}^n$ orthogonal to each and every vector in $row(A)$" : in set notation this is

$$(row(A))^{\perp} := \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{r} \cdot \mathbf{x} = 0 \text{ for every } \mathbf{r} \in row(A)\}. \tag{75}$$

So with (73) and (69) is not hard to show that $\left(row(A)\right)^{\perp} = nul(A)$: Since by (69)

$$\mathbf{r} \in row(A) := Span\left\{\mathbf{r}_1, \ldots, \mathbf{r}_m\right\} := \left\{c_1\mathbf{r}_1 + \ldots + c_m\mathbf{r}_m \middle| c_1, \ldots, c_m \in \mathbb{R}\right\}, \tag{76}$$

we can rewrite (75) as

$$\begin{aligned}\left(row(A)\right)^{\perp} &:= \left\{\mathbf{x} \in \mathbb{R}^n \middle| \left(c_1\mathbf{r}_1 + \ldots + c_m\mathbf{r}_m\right)\cdot\mathbf{x} = 0 \text{ for all } c_1, \ldots, c_m \in \mathbb{R}\right\} \\ &= \left\{\mathbf{x} \in \mathbb{R}^n \middle| c_1\mathbf{r}_1 \cdot \mathbf{x} + \ldots + c_m\mathbf{r}_m \cdot \mathbf{x} = 0 \text{ for all } c_1, \ldots, c_m \in \mathbb{R}\right\}.\end{aligned} \tag{77}$$

Thus

$$\mathbf{x} \in \left(row(A)\right)^{\perp} \Leftrightarrow c_1\mathbf{r}_1 \cdot \mathbf{x} + \ldots + c_m\mathbf{r}_m \cdot \mathbf{x} = 0 \text{ for all } c_1, \ldots, c_m \in \mathbb{R}, \tag{78}$$

(provided $\mathbf{x} \in \mathbb{R}^n$ so that the dot product makes sense). But now we see that the right hand sides of (73) and (78) are equivalent, hence the left hand sides are (and, so, $nul(A) = \left(row(A)\right)^{\perp}$, as advertized): we have

$$\mathbf{r}_1 \cdot \mathbf{x} = \ldots = \mathbf{r}_m \cdot \mathbf{x} = 0 \Rightarrow c_1\mathbf{r}_1 \cdot \mathbf{x} + \ldots + c_m\mathbf{r}_m \cdot \mathbf{x} = c_1 0 + \ldots + c_m 0 = 0 \tag{79}$$

for any and all $c_1, \ldots, c_m \in \mathbb{R}$, and conversely if

$$c_1\mathbf{r}_1 \cdot \mathbf{x} + \ldots + c_m\mathbf{r}_m \cdot \mathbf{x} = 0 \text{ for all } c_1, \ldots, c_m \in \mathbb{R} \tag{80}$$

then $\mathbf{r}_1 \cdot \mathbf{x} = \ldots = \mathbf{r}_m \cdot \mathbf{x} = 0$—choose all $c's$ to be zero except the first, then all to be zero except the second, etc., etc.

Whew! We just proved

$$\left(row(A)\right)^{\perp} = nul\left(A\right) \tag{81}$$

for any matrix A. Note with A being $m \times n$, each side of (81) is a subspace of $\mathbb{R}^n$.

Recently we proved "the dimension theorem", which, with A being $m \times n$ as above, says that

$$\dim col(A) + \dim nul(A) = n = \# \text{ of columns of } A = \dim \mathbb{R}^n. \tag{82}$$

The proof of this was basically the following: the theorem is obviously true if A is already in reduced row echelon form R—basically just that pivot columns plus nonpivot columns= total number of columns—and that even when

$A \neq rref(A) =: R$, we nevertheless get $nul(A) = nul(R)$ (whence $\dim nul(A) = \dim nul(R)$), and also manage to get $\dim col(A) = \dim col(R)$ (even though it may be that $col(A) \neq col(R)$, which is usually the case). Unfortunately (82) does ***not*** imply anything like

$$col(A_{mxn}) \bigcup nul(A_{mxn}) = \mathbb{R}^n, \tag{83}$$

and this simply because $col(A_{mxn}) \underset{\text{space}}{\subset} \mathbb{R}^m \neq \mathbb{R}^n$. On the other hand, we know by theorem that $\dim col(A) = \dim row(A)$, so that (82) could have been written as

$$\dim row(A) + \dim nul(A) = \dim \mathbb{R}^n, \tag{84}$$

or, with our new result (81), this could have been written as

$$\dim row(A) + \dim \left(row(A)\right)^{\perp} = \dim \mathbb{R}^n. \tag{85}$$

Now each of the vector spaces indicated in (85) ***is*** a subspace of $\mathbb{R}^n$ (for $A = A_{mxn}$), and indeed it's not too hard to prove then (***using*** (85)) that a)

$$row(A) + \left(row(A)\right)^{\perp} = \mathbb{R}^n \tag{86}$$

and b)

$$row(A) \bigcap \left(row(A)\right)^{\perp} = \left\{\mathbf{0}_n\right\} \subset \mathbb{R}^n. \tag{87}$$

In (86) the sum symbol $+$, which "adds" the sets, means that we form a new set that arises from taking every vector in $row(A)$ and vector adding it to every vector in $\left(row(A)\right)^{\perp}$. So this includes the union $row(A) \bigcup \left(row(A)\right)^{\perp}$ of the two sets, but can be a lot bigger.

And ***then*** , what's even cooler, because of (86) and (87) we rather immediately get that, given any $m \times n$ matrix A, $\mathbb{R}^n$ is "uniquely decomposed" into A's row space and its orthogonal complement—which means that, for any $\mathbf{v} \in \mathbb{R}^n$, there's a unique $\mathbf{r} \in row(A)$ and a unique $\mathbf{r}^{\perp} \in \left(row(A)\right)^{\perp} = nul(A)$ such that

$$\mathbf{v} = \mathbf{r} + \mathbf{r}^{\perp}. \tag{88}$$

Ultimately, this decomposition theorem is central to lots of other ideas in linear algebra, including the theory behind "regression", "least squares", etc. (the latter coming explicitly later in the course). Finally, because of (88), it's not too hard to show that

$$\left((row(A))^{\perp}\right)^{\perp} = row(A)\,. \tag{89}$$

(89) says "The orthogonal complement of the orthogonal complement of the row space of a matrix is the (original) row space of that matrix." This theorem seems rather obvious, and indeed its super easy to show that

$$\left((row(A))^{\perp}\right)^{\perp} \supset row(A) \tag{90}$$

(which says that ever vector $\mathbf{r}$ in $row(A)$ is orthogonal to every vector $\mathbf{r}^{\perp}$ that's orthogonal to every vector $\mathbf{r}'$ in $row(A)$), but much harder to show that

$$\left((row(A))^{\perp}\right)^{\perp} \subset row(A) \tag{91}$$

(which says that every vector $\mathbf{r}^{\perp\perp}$ that's orthogonal to every vector $\mathbf{r}^{\perp}$ that's orthogonal to every vector $\mathbf{r}$ in $row(A)$ is in $row(A)$). Indeed showing the unique decomposition (88) is central to getting this tough last part. I'll even show you below that this last part—the inclusion $\left((row(A))^{\perp}\right)^{\perp} \subset row(A)$ — is ***not*** true with a "slight" tweak to the problem (which shows that there's something highly nontrivial about getting the latter when we manage to get it).

Even cooler than any of the above is the following: who says you need to be thinking about matrices? After all, the row space of "the matrix" was just all linear combinations of a set of however many vectors ("m"of them) from $\mathbb{R}^n$ — who says we need to "stick them in a matrix"—and then $(row(A))^{\perp}$ is just the orthogonal complement of that subspace. So ultimately we have the following "matrix-free" theorem (proved ultimately by thinking about matrices as above):

THEOREM SUPER: Let W be a subspace of $\mathbb{R}^n$, and let $W^{\perp}$ be W's orthogonal complement subspace (in $\mathbb{R}^n$). Then

a) $\dim W + \dim W^{\perp} = \dim \mathbb{R}^n = n$,
b) $W \bigcup W^{\perp} = \mathbb{R}^n$
c) $W \bigcap W^{\perp} = \{\mathbf{0}_n\} \subset \mathbb{R}^n$
d) For every $\mathbf{v} \in \mathbb{R}^n$ there is a unique $\mathbf{w} \in W$ and a unique $\mathbf{w}^{\perp} \in W^{\perp}$ such that $\mathbf{v} = \mathbf{w} + \mathbf{w}^{\perp}$.
e) $\left(W^{\perp}\right)^{\perp} = W$.

We can get even cooler/more general still: by introducing coordinate vectors, one can imagine that the concrete vector spaces in the above theorem are simply spans of ***coordinate vectors*** associated with a basis—a basis for an ***abstract*** vector space. Of course we can't just have a vector space, but rather we need an inner product space (so that there's a notion of innerproduct hence orthogonality). Thus the coolest theorem so far that follows rather immediately from "dull" theorem (85)/(86)/(87)/(88) is the following:

THEOREM SUPER DUPER: Let W be a subspace of finite-dimensional innerproduct space V, and let $W^{\perp}$ be W's orthogonal complement subspace (with respect to V). Then

a) $\dim W + \dim W^{\perp} = \dim V$
b) $W \bigcup W^{\perp} = V$
c) $W \bigcap W^{\perp} = \{\mathbf{z}\} \subset V$
d) For every $\mathbf{v} \in V$ there is a unique $\mathbf{w} \in W$ and a unique $\mathbf{w}^{\perp} \in W^{\perp}$ such that $\mathbf{v} = \mathbf{w} + \mathbf{w}^{\perp}$.
e) $\left(W^{\perp}\right)^{\perp} = W$.

As in (90), for e) above it's easy to show that $\left(W^{\perp}\right)^{\perp} \supset W$, rather hard to show $\left(W^{\perp}\right)^{\perp} \subset W$. Indeed, as suggested above, it's easy to change the hypothesis of THEOREM SUPER DUPER and make it so that $\left(W^{\perp}\right)^{\perp} \not\subset W$. How? Just let V be infinite dimensional. Is this a 'tweak'? You decide. Anyways, who cares? Well, lot's of practitioners care, namely all thoughtful engineers/physicists: the vector spaces of the sciences are almost always infinite dimensional. :/

Anyways,….
You're task, should you choose to accept it, is to prove that, given a) (and c) of THEOREM SUPER (not THEOREM SUPER DUPER), you get b) and d). (Note that e) follows rather immediately from d), but we'll hold off on that for now—let you do it for extra credit say.) I'll get you started on proving b) and d) given a) and c).

Well, let's first get c) easily: first, we certainly have

$$\mathbf{v} \in W \bigcap W^{\perp} \Leftrightarrow \mathbf{v} \in W \ \text{ and } \mathbf{v} \in W^{\perp} \tag{92}$$

where

$$W^{\perp} := \left\{ \mathbf{w}^{\perp} \in \mathbb{R}^n \middle| \mathbf{w}^{\perp} \cdot \mathbf{v} = 0 \ \text{ for every } \mathbf{v} \in W \right\}. \tag{93}$$

So since (92) says $\mathbf{v} \in W \bigcap W^{\perp}$ is in both sets we get

$$\mathbf{v} \in W \bigcap W^{\perp} \Rightarrow 0 = \mathbf{v} \cdot \mathbf{v} = \|\mathbf{v}\|^2 \Leftrightarrow \mathbf{v} = \mathbf{0}_n \in \mathbb{R}^n \tag{94}$$

and we're done. (In (94) we used that the Euclidean innerproduct—the "dot product" –is "nondegenerate", i.e., the only vector of length zero is the zero vector.

To prove b) then d) do the following: First, if $W = \{\mathbf{0}_n\}$ then by (93) one easily gets $W^{\perp} = \mathbb{R}^n$ and, so, gets $W \bigcup W^{\perp} = \{\mathbf{0}_n\} \bigcup \mathbb{R}^n = \mathbb{R}^n$, which is b) , and easily gets $\mathbf{w} \in W = \{\mathbf{0}_n\}$ and $\mathbf{w}^{\perp} \in W^{\perp} = \mathbb{R}^n$ adding up to any $\mathbf{v} \in \mathbb{R}^n$ are unique (which is d)) since then $\mathbf{v} = \mathbf{w} + \mathbf{w}^{\perp} = \mathbf{0}_n + \mathbf{w}^{\perp} = \mathbf{w}^{\perp}$. Likewise if $W = \mathbb{R}^n$, then from (93) one easily gets $W^{\perp} = \{\mathbf{0}_n\}$ (as in (94)) and, so, b) and d) follow again (in almost exactly the same way). So now that we've got the two trivial cases out of the way, let's do "everything in between":

Since W and $W^{\perp}$ are (finite dimensional) vector spaces, and since we now preclude either one of them from being only the zero vector space (we handled those two cases above), they each have a basis: let

$$S_W = \{\mathbf{w}_1, \ldots, \mathbf{w}_l\} \tag{95}$$

and

$$S_{W^{\perp}} = \{\mathbf{u}_1, \ldots, \mathbf{u}_k\} \tag{96}$$

be bases for W and $W^{\perp}$, respectively. But from a) we have

$$n = \dim \mathbb{R}^n = \dim W + \dim W^{\perp} = l + k. \tag{97}$$

Of course $l + k$ is the cardinality of the union of the two sets: there's l elements in S_W , k of them in $S_{W^{\perp}}$, and by c) the only thing they could have in common is the zero vector, which they can't have in common since otherwise they'd both be linearly dependent. So

$$S := S_W \bigcup S_{W^{\perp}} = \{\mathbf{w}_1, \ldots, \mathbf{w}_l, \mathbf{u}_1, \ldots, \mathbf{u}_k\} \subset \mathbb{R}^n, \tag{98}$$

and

$$|S| = l + k = n = \dim \mathbb{R}^n. \tag{99}$$

So here's a set of n vectors in $\mathbb{R}^n$, which then, theorem, will be a basis for $\mathbb{R}^n$ provided, say, the set is linearly independent. This will end up being the case—i.e., S will end up being a basis for $\mathbb{R}^n$, and, so, b) will follow since S will ***span*** $\mathbb{R}^n$, and d) will follow since the overall coordinate vector $(\mathbf{v})_S$ of any $\mathbf{v} \in \mathbb{R}^n$ with respect to basis S will be unique. (Recall the uniqueness of coordinate vectors with respect to a basis follows from the ***independence*** of a basis.) So the only thing I'm asking you to do is to prove that S defined by (98) is linearly independent. Oh, ok, I'll get you started on that too: by definition, S of (98) is linearly independent iff

$$\alpha_1\mathbf{w}_1 + \ldots + \alpha_l\mathbf{w}_l + \beta_1\mathbf{u}_1 + \ldots + \beta_k\mathbf{u}_k = \mathbf{0}_n \Rightarrow \alpha_1 = \ldots = \alpha_l = \beta_1 = \ldots = \beta_k = 0. \quad (100)$$

Well, consider the first equation in (100) and rewrite it as

$$\alpha_1\mathbf{w}_1 + \ldots + \alpha_l\mathbf{w}_l = (-\beta_1)\mathbf{u}_1 + \ldots + (-\beta_k)\mathbf{u}_k. \quad (101)$$

Hmm…. On the left of (101) we have a linear combination of vectors in W— whence, by closure under linear combos, the *left hand side* of (101) is a vector in W, and on the right of (101) we have a linear combination of vectors in $W^\perp$— whence, by closure under linear combos, the *right hand side* of (101) is a vector in $W^\perp$. See where this is going? I hope so—I've taken you to the 99meter line in the 100meter sprint. Well, I've taken you at least that far if I now remind you that since $S_W = \{\mathbf{w}_1, \ldots, \mathbf{w}_l\}$ is linearly independent and since $S_{W^\perp} = \{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ is linearly independent then we have both

$$\alpha_1\mathbf{w}_1 + \ldots + \alpha_l\mathbf{w}_l = \mathbf{0}_n \Rightarrow \alpha_1 = \ldots = \alpha_l = 0, \quad (102)$$

and

$$(-\beta_1)\mathbf{u}_1 + \ldots + (-\beta_k)\mathbf{u}_k = \mathbf{0}_n \Rightarrow -\beta_1 = \ldots = -\beta_k = 0. \quad (103)$$

<u>10pts</u>

<u>Solution</u>

Since each side of (101) is in both W and $W^\perp$, then each side is in $W \cap W^\perp = \{\mathbf{0}_n\}$ (recall (94)) , i.e., each side is $\mathbf{0}_n$ and we immediately get the left hand sides of both (102) and (103), whence both the right hand sides of (102) and (103), which gives the right hand side of (100) and we're done.

11) Find the standard matrix for the following linear operator on $\mathbb{R}^3$: A rotation of 180° counter clockwise about the z axis, followed by a rotation of 90° counter clockwise about the y axis, followed by a rotation of 270° counter clockwise about the x axis.

10pts

Solution

By theorem we have that for linear operator $T:\mathbb{R}^3 \to \mathbb{R}^3$

$$[T] = \left[T(\hat{x})\middle|T(\hat{y})\middle|T(\hat{z})\right] = \left[T\begin{pmatrix}1\\0\\0\end{pmatrix}\middle|T\begin{pmatrix}0\\1\\0\end{pmatrix}\middle|T\begin{pmatrix}0\\0\\1\end{pmatrix}\right] \tag{104}$$

Now $\hat{x} = (1,0,0)^T$ is sent to $-\hat{x} = (-1,0,0)^T$ by the 180° counter clockwise rotation about the z axis, and the rotation of 90° counter clockwise about the y axis sends it to $\hat{z} = (0,0,1)^T$, and then the rotation of 270° counter clockwise about the x axis sends this to $\hat{y} = (0,1,0)^T$. Similarly, $\hat{y} = (0,1,0)^T$ is changed to $-\hat{y} = (0,-1,0)^T$ by the 180° counter clockwise rotation about the z axis, and the latter is unchanged by a rotation about the y axis, which then goes to $\hat{z} = (0,0,1)^T$ via a rotation of 270° counter clockwise about the x axis. Finally $\hat{z} = (0,0,1)^T$ is unchanged by a rotation about its axis, which then is changed to $\hat{x} = (1,0,0)^T$ by a rotation of 90° counter clockwise about the y axis, which then is fixed by any rotation by that axis. Thus,

$$[T] = \left[T(\hat{x})\middle|T(\hat{y})\middle|T(\hat{z})\right] = \left[\hat{y}\middle|\hat{z}\middle|\hat{x}\right] = \begin{bmatrix}0&0&1\\1&0&0\\0&1&0\end{bmatrix}. \tag{105}$$

If these operations are composed in the opposite order, one would get the matrix

$$\begin{bmatrix}0&1&0\\0&0&-1\\-1&0&0\end{bmatrix},$$

which is incorrect.

12) Let $T:\mathbb{R}^3 \to \mathbb{R}^2$ be linear. Show that T is not one-to-one. (Hint: Cook this down to a homogeneous linear system with more unknowns than equations.)

10pts

Solution

Be definition, T is one-to-one iff

$$T(\mathbf{x}) = T(\mathbf{y}) \Rightarrow \mathbf{x} = \mathbf{y} \tag{106}$$

and, easy theorem, when T is linear (106) is equivalent to

$$T(\mathbf{x}) = \mathbf{0}_2 \in \mathbb{R}^2 \Rightarrow \mathbf{x} = \mathbf{0}_3 \in \mathbb{R}^3. \tag{107}$$

But since

$$\mathbf{x} \in \mathbb{R}^3 \Leftrightarrow \exists x_1, x_2, x_3 \in \mathbb{R} \text{ such that } \mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3, \tag{108}$$

where $S := \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is the standard basis for $\mathbb{R}^3$, then, with linearity again, which dictates that

$$\begin{aligned} T(x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3) &= x_1T(\mathbf{e}_1) + x_2T(\mathbf{e}_2) + x_3T(\mathbf{e}_3) = \left[T(\mathbf{e}_1)\middle| \quad T(\mathbf{e}_2)\middle| \quad T(\mathbf{e}_3)\right]\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \\ &=: [T]\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} =: [T]\mathbf{x} = [T]_{2\times3}\,\mathbf{x}, \end{aligned} \tag{109}$$

we get (107) is

$$[T]_{2\times3}\,\mathbf{x} = \mathbf{0}_2 \in \mathbb{R}^2 \Rightarrow \mathbf{x} = \mathbf{0}_3 \in \mathbb{R}^3, \tag{110}$$

where, by (109) (and $T : \mathbb{R}^3 \to \mathbb{R}^2$) , and as indicated, we see that $[T]_{2\times3}$ is a 2×3 matrix. So the equation $[T]_{2\times3}\,\mathbf{x} = \mathbf{0}_2$ is a linear homogeneous system of 2 equations with 3 unknowns and, so, has more than just the trivial solution. Thus (110) does not hold, equivalently (107) does not hold, and T is not one-to-one, as advertized.

13) Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be linear. Show that T is not onto. (Hint: Cook this down to using that a set of fewer than n vectors cannot span an n dimensional vector space.)

10pts

Solution

$T:\mathbb{R}^2 \to \mathbb{R}^3$ is onto ($\mathbb{R}^3$) iff for every $\mathbf{b} \in \mathbb{R}^3$ there is an $\mathbf{x} \in \mathbb{R}^2$ such that

$$T(\mathbf{x}) = \mathbf{b}. \tag{111}$$

Picking as above the standard basis, but this time for $\mathbb{R}^2$, (111) is equivalent to

$$[T]_{3\times 2}\,\mathbf{x} = \mathbf{b} \tag{112}$$

since T is linear. So our linear $T:\mathbb{R}^2 \to \mathbb{R}^3$ is onto iff (112) has a solution $\mathbf{x} \in \mathbb{R}^2$ for every $\mathbf{b} \in \mathbb{R}^3$. But since

$$[T]_{3\times 2}\,\mathbf{x} = \left[T(\mathbf{e}_1)\middle|T(\mathbf{e}_2)\right]_{3\times 2}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 T(\mathbf{e}_1) + x_2 T(\mathbf{e}_2) \in span\{T(\mathbf{e}_1), T(\mathbf{e}_2)\}, \tag{113}$$

that would say that every $\mathbf{b} \in \mathbb{R}^3$ is a linear combination in the set $\{T(\mathbf{e}_1), T(\mathbf{e}_2)\}$, i.e., it would say that $\{T(\mathbf{e}_1), T(\mathbf{e}_2)\}$ spans a three dimensional space. This contradicts the theorem that a set of fewer than n vectors cannot span an n dimensional vector space. Thus (112) can't have a solution $\mathbf{x}$ for every $\mathbf{b}$, likewise for (111), and T is not onto, as advertized.

14) Let $T:\mathbb{R}^2 \to \mathbb{R}^2$ be linear, and let T be onto. Show that T is one-to-one. (Hint: Cook this down to the fact that a set of n vectors in an n dimensional vector space is a basis if it spans the space.)

10pts

Solution

As in (107) we'll get that the map is one-to-one iff the only pre-image of $\mathbf{0}_2 \in \mathbb{R}^2$ is $\mathbf{0}_2 \in \mathbb{R}^2$. But by definition of linear independence, and by the way matrix multiplication works (as in (109) or (113)) that will hold iff the columns of $[T]_{2\times 2}$ are linearly independent. But those columns are a set of 2 vectors in 2 dimensional vector space $\mathbb{R}^2$, and, so, will form a basis iff they span the space. Well, they do span the space: the range of $[T]_{2\times 2}$ is it's columns space, and we've said T is onto, i.e., we've said $[T]_{2\times 2}$'s column space is all of $\mathbb{R}^2$. Thus those column vectors are a basis and, in particular, are linearly independent.

15) Let $T:\mathbb{R}^2 \to \mathbb{R}^2$ be linear, and let T be one-to-one. Show that T is onto. (Hint: Cook this down to the fact that a set of n vectors in an n dimensional vector space is a basis if it's linearly independent.)

10pts

Solution

As above, by definition of onto, we'll conclude T is onto if $[T]_{2\times 2}$'s columns span $\mathbb{R}^2$, which will occur if they're independent (by the "right number of vectors in the right-sized space" theorem), which happens here since T is one-to-one.