Math 313 Final KEY
Spring 2010, June 17
section 001
Instructor: Scott Glasgow

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Signature

1) Show that if a system of linear equations

$$A\mathbf{x} = \mathbf{b} \tag{1}$$

has 2 (distinct) solutions $\mathbf{x}$ then it also has infinitely many (distinct) solutions. (Do not assume the theorem that a linear system has only 0, 1, or infinitely many solutions. Rather use matrix algebra to show this must be the case.)

15pts

Solution

If $\mathbf{x}_1$ and $\mathbf{x}_2 \neq \mathbf{x}_1$ both solve (1), then for each $t \in \mathbb{R}$, ***of which there are infinitely many***, we get (using matrix algebra) that a simple interpolation "between" these solutions gives

$$A\left(\left(1-t\right)\mathbf{x}_1 + t\mathbf{x}_2\right) = \left(1-t\right)A\mathbf{x}_1 + tA\mathbf{x}_2 = \left(1-t\right)\mathbf{b} + t\mathbf{b} = \left(\left(1-t\right)+t\right)\mathbf{b} = 1\mathbf{b} = \mathbf{b}\,, \tag{2}$$

i.e. there are infinitely many distinct solution solutions of (1) (of the form $\mathbf{x} = \left(1-t\right)\mathbf{x}_1 + t\mathbf{x}_2 = \mathbf{x}_1 + t\left(\mathbf{x}_2 - \mathbf{x}_1\right)$). Note that it is important that $\mathbf{x}_2 \neq \mathbf{x}_1$ to get this result as otherwise $\left(1-t\right)\mathbf{x}_1 + t\mathbf{x}_2 = \left(\left(1-t\right)+t\right)\mathbf{x}_1 = 1\mathbf{x}_1 = \mathbf{x}_1$ only represents 1 solution, not infinitely many.

2) By using the (permutation) definition of the determinant, prove that if square matrix B is the same as square matrix A, except that one of B's rows is a scalar multiple k of the corresponding row of A, then

$$\det B = k \det A. \tag{3}$$

15 points

Solution

Let it be the j^{th} row of the matrix B that is the same as the scalar k multiplied by the j^{th} row of the matrix A. Then, by definition of the determinant, and the relationship between the 2 rows (of matrix A and matrix B), we have

$$\begin{aligned}\det B &:= \sum_{p:[n]\to[n]} \left(-1\right)^{d(p)} b_{1p(1)} \cdots b_{jp(j)} \cdots b_{np(n)} = \sum_{p:[n]\to[n]} \left(-1\right)^{d(p)} a_{1p(1)} \cdots \left(ka_{jp(j)}\right) \cdots a_{np(n)} \\ &= k \sum_{p:[n]\to[n]} \left(-1\right)^{d(p)} a_{1p(1)} \cdots a_{jp(j)} \cdots a_{np(n)} =: k \det A.\end{aligned} \tag{4}$$

3) Show that if a matrix A is invertible, the system $A\mathbf{x} = \mathbf{b}$ has one and only one solution $\mathbf{x}$, namely $\mathbf{x} = A^{-1}\mathbf{b}$. [Warning: there are two things to prove here, namely a) that ***if*** the system has a solution, then it can only be $\mathbf{x} = A^{-1}\mathbf{b}$, and that b) $\mathbf{x} = A^{-1}\mathbf{b}$ actually does solve the system. Here then you will have addressed the "one and only one" issues in reverse order: you may first show that a) there is at most one solution, and b) that there is in fact one solution (rather than none). In parts a) and b) you will use that A^{-1} is A's left and right inverse, ***respectively***.]

20 points

Solution

If the system has a solution $\mathbf{x}$, then, for any such $\mathbf{x}$, we may certainly write

$$A\mathbf{x} = \mathbf{b} \tag{5}$$

without implicitly lying, and then, by left application of A^{-1} to (5), as well as by the associative property of matrix multiplication, obtain that

$$\mathbf{x} = I\mathbf{x} = \left(A^{-1}A\right)\mathbf{x} = A^{-1}\left(A\mathbf{x}\right) = A^{-1}\mathbf{b}\ . \tag{6}$$

In (6) we also used that A^{-1} is a left inverse of A, as well as the fact that the so-called identity matrix I is in fact a "multiplicative identity". Here then we have just showed that if (5) has a solution, it's got to be $\mathbf{x} = A^{-1}\mathbf{b}$. Thus we have showed that (5) has at most one solution. But our demonstration does not yet preclude there being no solution. To preclude that possibility, we confirm that, for the only promising candidate, namely $\mathbf{x} = A^{-1}\mathbf{b}$, we get

$$A\mathbf{x} = A\left(A^{-1}\mathbf{b}\right) = \left(AA^{-1}\right)\mathbf{b} = I\mathbf{b} = \mathbf{b}, \tag{7}$$

so that our candidate was successful. (Here we have used the associative property of matrix multiplication, the fact that A^{-1} is a right inverse of A, as well as the fact that the so-called identity matrix I is in fact a "multiplicative identity".) Thus we have showed that the system has one and only one solution, namely $\mathbf{x} = A^{-1}\mathbf{b}$.

4) For the given set of objects, together with the indicated notions of addition and scalar multiplication, determine whether each of the ten vector space axioms holds: real pairs (x, y), where

$$(x,y)+(x',y') := (x+x', y+y'), \quad k(x,y) := \left(\frac{kx+2ky}{5}, \frac{2kx+4ky}{5}\right). \tag{8}$$

30pts

Solution

1) through 5) : Since $V = \mathbb{R}^2$ but with only scalar multiplication differing, these axioms hold (since they reference only vector addition).

6) $k(x,y) \in V$ when $(x,y) \in V$ and $k \in \mathbb{R}$ since both $\frac{kx+2ky}{5}$ and $\frac{2kx+4ky}{5}$ are clearly real numbers then.

7) We have

$$k\left((x,y)+(w,z)\right) = k\left(x+w, y+z\right) = \left(\frac{k(x+w)+2k(y+z)}{5}, \frac{2k(x+w)+4k(y+z)}{5}\right)$$

$$= \left(\frac{kx+2ky}{5}+\frac{kw+2kz}{5}, \frac{2kx+4ky}{5}+\frac{2kw+4kz}{5}\right)$$

$$= \left(\frac{kx+2ky}{5}, \frac{2kx+4ky}{5}\right)+\left(\frac{kw+2kz}{5}, \frac{2kw+4kz}{5}\right)$$

$$= k(x,y)+k(w,z),$$

so this axiom holds.

8) We have

$$(k+m)(x,y) = \left(\frac{(k+m)x+2(k+m)y}{5}, \frac{2(k+m)x+4(k+m)y}{5}\right)$$

$$= \left(\frac{kx+2ky}{5}+\frac{mx+2my}{5}, \frac{2kx+4ky}{5}+\frac{2mx+4my}{5}\right)$$

$$= \left(\frac{kx+2ky}{5}, \frac{2kx+4ky}{5}\right)+\left(\frac{mx+2my}{5}, \frac{2mx+4my}{5}\right)$$

$$= k(x,y)+m(x,y),$$

so this axiom holds

9) We have

$$
\begin{aligned}
k\left(m(x,y)\right) &= k\left(\frac{mx+2my}{5},\frac{2mx+4my}{5}\right) \\
&= \left(\frac{k\left(\frac{mx+2my}{5}\right)+2k\left(\frac{2mx+4my}{5}\right)}{5},\frac{2k\left(\frac{mx+2my}{5}\right)+4k\left(\frac{2mx+4my}{5}\right)}{5}\right) \\
&= \left(\frac{\frac{5kmx}{5}+\frac{2\cdot 5kmy}{5}}{5},\frac{\frac{2\cdot 5kmx}{5}+\frac{4\cdot 5kmy}{5}}{5}\right) \\
&= \left(\frac{(km)x+2(km)y}{5},\frac{2(km)x+4(km)y}{5}\right) = (km)(x,y),
\end{aligned}
\tag{9}
$$

so this axiom holds.

10) We have

$$
1(x,y) := \left(\frac{1\cdot x+2\cdot 1\cdot y}{5},\frac{2\cdot 1\cdot x+4\cdot 1\cdot y}{5}\right) = \left(\frac{x+2y}{5},\frac{2x+4y}{5}\right)
$$

which is not (x,y) in every instance. For example, $1(2,-1) = \left(\frac{2+2\cdot(-1)}{5},\frac{2\cdot 2+4\cdot(-1)}{5}\right) = (0,0) \neq (2,-1)$. Thus all axioms hold except the last.

5) Prove that for any (real) vector space $(V,\mathbb{R},+,\bullet)$ (satisfying the ten axioms)—no matter how bizarre the addition $+$ and the scalar multiplication $\bullet$ are—we must have $0\bullet\mathbf{u} = \mathbf{z}$ for any vector $\mathbf{u} \in V$ ($\in \mathbb{R}$), where $\mathbf{z}$ is the "zero" vector in the space, i.e. where $\mathbf{z}$ is the additive identity in V. Be sure to list the axioms used in your proof. Feel free to use the fact that

$$
\begin{aligned}
&\mathbf{w}+\mathbf{v}=\mathbf{v} \Rightarrow \mathbf{w}=\mathbf{z}, \text{ or} \\
&\mathbf{v}+\mathbf{w}=\mathbf{v} \Rightarrow \mathbf{w}=\mathbf{z},
\end{aligned}
\tag{10}
$$

i.e. that if a vector $\mathbf{w}$ acts like $\mathbf{z}$ even for just one $\mathbf{v} \in V$, then it is $\mathbf{z}$. On the other hand, you may also do what you did in the relevant type of homework problems (which invents the fact indicated in equation (10) for you).

<u>20pts</u>

Solution

By axiom 8)

$$0\bullet\mathbf{u}+0\bullet\mathbf{u}=\left(0+0\right)\bullet\mathbf{u}, \tag{11}$$

which, by property of the number $0\in\mathbb{R}$, gives

$$0\bullet\mathbf{u}+0\bullet\mathbf{u}=0\bullet\mathbf{u}. \tag{12}$$

But now this is the left-hand side of equation (10) above with $\mathbf{w}=0\bullet\mathbf{u}$ (and, less important, $\mathbf{v}=0\bullet\mathbf{u}$). So by the right-hand side of equation (10) $\mathbf{w}=0\bullet\mathbf{u}=\mathbf{z}$.

6) By use of the relevant "if and only if" theorem, determine whether the following is a subspace of M_{nn}: (M_{nn} is the vector space of $n\times n$ matrices with ordinary matrix addition and scalar multiplication.) the set W of all $n\times n$ matrices A such that $A^T=-A$. MAKE SURE AND REFERENCE AND **USE** THE THEOREM in determining your conclusion. Either way, prove your conclusion. Note that we have chosen here

$$W=\left\{A\in M_{nn}\middle|A^T=-A\right\}. \tag{13}$$

In referencing the theorem, it might be helpful to refer to W .

30pts

Solution

The theorem is as follows: Let W be a non-empty ***subset*** of elements of a real vector space V . Then W is, in addition, a (real) ***subspace*** of V iff

$$c,k\in\mathbb{R},\ \mathbf{u},\mathbf{v}\in W\Rightarrow c\mathbf{u}+k\mathbf{v}\in W. \tag{14}$$

Changing the notation in (14) to be more traditional for matrices (as in (13)) we could write (14) as

$$c,k\in\mathbb{R},\ A,A'\in W\Rightarrow cA+kA'\in W. \tag{15}$$

To check that our hypotheses hence conclusion of this theorem hold, we first note that the subset W defined by (13) is nonempty: if nothing else $A=0$ (the zero matrix) is "skew", i.e. satisfies $A^T=-A$, so that $W\supset\left\{0\in M_{nn}\right\}\neq\varnothing$, where $\varnothing$ is notation for the empty set. (A square zero matrix is also symmetric, i.e. $A^T=A$ for $A=0\in M_{nn}$, but this is not relevant.) To show (15) always holds and, so, to show the set W is actually a subspace of M_{nn} we note that when A and A' are both in subset W (giving both $A^T = -A$ and $A'^T = -A'$), and when c, k are arbitrary real numbers, we have $cA + kA'$ is also in W because

$$\left(cA + kA'\right)^T = cA^T + kA'^T = c\left(-A\right) + k\left(-A'\right) = -\left(cA + kA'\right). \tag{16}$$

In (16) we used, in order, properties of the transpose, membership of A and A' in subset W and finally properties of matrix algebra.

7) Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ be a basis for a vector space V. It turns out that $S' = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is also a basis for V, where

$$\mathbf{u}_1 = \mathbf{v}_1, \quad \mathbf{u}_2 = \mathbf{v}_1 + \mathbf{v}_2, \quad \mathbf{u}_3 = \mathbf{v}_1 + 2\mathbf{v}_2 + \mathbf{v}_3. \tag{17}$$

So suppose the coordinate vector $(\mathbf{v})_S$ of a vector $\mathbf{v} \in V$ relative to basis S is given by

$$(\mathbf{v})_S = (a, b, c). \tag{18}$$

What is the coordinate vector $(\mathbf{v})_{S'}$ of $\mathbf{v}$ relative to basis S'?

20pts

Solution

By definition of coordinate vector, (18) holds with $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ iff

$$\mathbf{v} = a\mathbf{v}_1 + b\mathbf{v}_2 + c\mathbf{v}_3. \tag{19}$$

(17) is inverted by noting then that

$$\begin{aligned} &\mathbf{v}_1 = \mathbf{u}_1, \quad \mathbf{v}_2 = \mathbf{u}_2 - \mathbf{v}_1 = -\mathbf{u}_1 + \mathbf{u}_2, \\ &\mathbf{v}_3 = \mathbf{u}_3 - \mathbf{v}_1 - 2\mathbf{v}_2 = \mathbf{u}_3 - \mathbf{u}_1 - 2\left(\mathbf{u}_2 - \mathbf{u}_1\right) = \mathbf{u}_1 - 2\mathbf{u}_2 + \mathbf{u}_3 \end{aligned} \tag{20}$$

i.e. (17) implies the inverse transformations

$$\mathbf{v}_1 = \mathbf{u}_1, \quad \mathbf{v}_2 = -\mathbf{u}_1 + \mathbf{u}_2, \quad \mathbf{v}_3 = \mathbf{u}_1 - 2\mathbf{u}_2 + \mathbf{u}_3. \tag{21}$$

Using (21) in (19) we get

$$\begin{aligned} \mathbf{v} &= a\mathbf{v}_1 + b\mathbf{v}_2 + c\mathbf{v}_3 = a\mathbf{u}_1 + b\left(-\mathbf{u}_1 + \mathbf{u}_2\right) + c\left(\mathbf{u}_1 - 2\mathbf{u}_2 + \mathbf{u}_3\right) \\ &= \left(a - b + c\right)\mathbf{u}_1 + \left(b - 2c\right)\mathbf{u}_2 + c\mathbf{u}_3. \end{aligned} \tag{22}$$

Thus, by definition,

$$(\mathbf{v})_{S'} = \left(a - b + c, b - 2c, c\right). \tag{23}$$

8) For the previous problem, what is the transition matrix $P_{SS'}$ ***from*** basis S' ***to*** basis S ? What is the transition matrix $P_{S'S}$ ***from*** basis S ***to*** basis S' ? Don't mix these two up. Maybe the easiest thing to do here is use the result of the last problem, but re-inventing can be useful as an independent check.

20pts

Solution

By definition, the matrix $P_{SS'}$ ***from*** basis S' ***to*** basis S behaves as

$$(\mathbf{v})_S = P_{SS'}(\mathbf{v})_{S'}, \tag{24}$$

where the coordinate vectors are obviously then column vectors. For problem 1) (24) is

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = P_{SS'} \begin{bmatrix} a-b+c \\ b-2c \\ c \end{bmatrix}. \tag{25}$$

But, by relevant transformations, (25) gives,

$$\begin{aligned} & P_{SS'} \begin{bmatrix} a-b+c \\ b-2c \\ c \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \Leftrightarrow P_{SS'} \begin{bmatrix} a \\ b-2c \\ c \end{bmatrix} = \begin{bmatrix} a+b-c \\ b \\ c \end{bmatrix} \\ & \Leftrightarrow P_{SS'} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a+b+c \\ b+2c \\ c \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix}. \end{aligned} \tag{26}$$

By making in turn choices of (a,b,c) corresponding to standard basis elements (of $\mathbb{R}^3$), on finds that we must have then that the columns of $P_{SS'}$ must be in turn the columns of the 3×3 matrix on the right of (26), i.e. $P_{SS'}$ must be that matrix: one finds

$$P_{SS'} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}. \tag{27}$$

To find $P_{S'S}$, we note by definition it behaves as

$$(\mathbf{v})_{S'} = P_{S'S}(\mathbf{v})_S, \tag{28}$$

which for problem 1) is evidently

$$P_{S'S} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a-b+c \\ b-2c \\ c \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix}, \tag{29}$$

and logic as above dictates

$$P_{S'S}=\begin{bmatrix}1 & -1 & 1\\ 0 & 1 & -2\\ 0 & 0 & 1\end{bmatrix}. \tag{30}$$

One checks the matrices of (27) and (30) are inverses, and that

$$\begin{gathered}P_{SS'}\begin{bmatrix}1\\0\\0\end{bmatrix}=\begin{bmatrix}1 & 1 & 1\\ 0 & 1 & 2\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1\\0\\0\end{bmatrix}=\begin{bmatrix}1\\0\\0\end{bmatrix},\quad P_{SS'}\begin{bmatrix}0\\1\\0\end{bmatrix}=\begin{bmatrix}1 & 1 & 1\\ 0 & 1 & 2\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}0\\1\\0\end{bmatrix}=\begin{bmatrix}1\\1\\0\end{bmatrix},\\ P_{SS'}\begin{bmatrix}0\\0\\1\end{bmatrix}=\begin{bmatrix}1 & 1 & 1\\ 0 & 1 & 2\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}0\\0\\1\end{bmatrix}=\begin{bmatrix}1\\2\\1\end{bmatrix},\end{gathered} \tag{31}$$

which patently agrees with (17). (And this indicates an alternate, perhaps better approach.)

9) Find the linear function $q(x)$ closest to the quadratic polynomial $p(x)=3x^2-3x+4$, where the notion of distance is given by

$$d^2(p,q)=\int_0^2 \left(p(x)-q(x)\right)^2 dx\,. \tag{32}$$

Feel free to use the fact that $S':=\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3\}:=\{1,x-1,3x^2-6x+2\}$ is an orthogonal basis for P_2 with respect to inner product $\langle,\rangle$ defined by

$$\langle p,q\rangle=\int_0^2 p(x)q(x)dx\,. \tag{33}$$

<u>20pts</u>

<u>Solution</u>

Since $p(x)=3x^2-3x+4=3x+2+1\cdot\left(3x^2-6x+2\right)$, and since $\mathbf{v}_3:=3x^2-6x+2\in\left(\text{Span}\{1,x\}\right)^{\perp}=\left(\text{Span}\{1,x-1\}\right)^{\perp}$ with respect to innerproduct (33), and since from (32) we have

$$d^2(p,q)=\int_0^2\left(p(x)-q(x)\right)^2 dx=\langle p-q,p-q\rangle, \tag{34}$$

the answer is, theorem,

$$q(x)=3x+2. \tag{35}$$

Alternatively, we have, theorem, (from the orthogonal basis $S' := \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} := \{1, x-1, 3x^2-6x+2\}$)

$$\begin{aligned}
q(x) &= \frac{\langle \mathbf{v}_1, p\rangle}{\langle \mathbf{v}_1, \mathbf{v}_1\rangle}\mathbf{v}_1(x) + \frac{\langle \mathbf{v}_2, p\rangle}{\langle \mathbf{v}_2, \mathbf{v}_2\rangle}\mathbf{v}_2(x) = \frac{\langle 1, 3x^2-3x+4\rangle}{2}\cdot 1 + \frac{\langle x-1, 3x^2-3x+4\rangle}{\frac{2}{3}}\cdot (x-1) \\
&= \frac{1}{2}\int_0^2 (3x^2-3x+4)\,dx + \frac{3}{2}\left(\int_0^2 (x-1)(3x^2-3x+4)\,dx\right)(x-1) \\
&= \frac{1}{2}\left(8 - \frac{3}{2}\cdot 4 + 8\right) + \frac{3}{2}\left(\int_0^2 (3x^3-6x^2+7x-4)\,dx\right)(x-1) \qquad (36) \\
&= 5 + \frac{3}{2}\left(\frac{3}{4}\cdot 16 - 2\cdot 8 + \frac{7}{2}\cdot 4 - 8\right)(x-1) \\
&= 5 + 3(x-1) = 3x+2,
\end{aligned}$$

as advertised.

10) Find the eigenvalues and eigen***<u>spaces</u>*** of linear operator $T : P_2 \to P_2$ defined by

$$T[f](x) = \frac{1}{8}\int_{-1}^{1}\left(-15 + 9x^2 + \left(45 - 15x^2\right)y^2\right) f(y)\,dy. \qquad (37)$$

(The "ugly" numbers here are designed to make the answers extremely simple; if you are not getting very simple eigenvalues and eigenspaces of functions, try again.)

<u>20pts</u>

<u>Solution</u>

By theorem, the eigenvalues of $T : P_2 \to P_2$ are those of any of its matrices $[T]_{SS}$, where we note that the only requirement is that the input basis S is the same as the output basis. Likewise the eigenvectors of $T : P_2 \to P_2$ are given by noting that their coordinate vectors with respect to basis S will be eigenvectors of matrix $[T]_{SS}$. So we start by finding $[T]_{SS}$, where, say, $S = \{1, x, x^2\} =: \{f_0(x), f_1(x), f_2(x)\}$ is the standard basis of P_2: since

$$
\begin{aligned}
T[f_0](x) &= \frac{1}{8}\int_{-1}^{1}\left(-15+9x^2+45y^2-15x^2y^2\right)f_0(y)dy = \frac{1}{8}\int_{-1}^{1}\left(-15+9x^2+45y^2-15x^2y^2\right)dy \\
&= \frac{2}{8}\left(-15+9x^2+\frac{45}{3}-\frac{15}{3}x^2\right) = x^2 = f_2(x) = 0\,f_0(x)+0\,f_1(x)+1\,f_2(x) \\
T[f_1](x) &= \frac{1}{8}\int_{-1}^{1}\left(-15+9x^2+45y^2-15x^2y^2\right)f_1(y)dy = \frac{1}{8}\int_{-1}^{1}\left(-15+9x^2+45y^2-15x^2y^2\right)ydy = 0 \\
&= 0\,f_0(x)+0\,f_1(x)+0\,f_2(x), \\
T[f_2](x) &= \frac{1}{8}\int_{-1}^{1}\left(-15+9x^2+45y^2-15x^2y^2\right)f_2(y)dy = \frac{1}{8}\int_{-1}^{1}\left(-15+9x^2+45y^2-15x^2y^2\right)y^2dy \\
&= \frac{1}{8}\int_{-1}^{1}\left(-15y^2+9x^2y^2+45y^4-15x^2y^4\right)dy \\
&= \frac{2}{8}\left(-\frac{15}{3}+\frac{9}{3}x^2+\frac{45}{5}-\frac{15}{5}x^2\right) = 1 = f_0(x) = 1\,f_0(x)+0\,f_1(x)+0\,f_2(x),
\end{aligned}
$$
(38)

and since, by definition, $[T]_{SS}$ behaves as

$$[T]_{SS}\,(f)_S = \left(T[f]\right)_S, \qquad (39)$$

then $[T]_{SS}$'s first column must be

$$[T]_{SS}\begin{bmatrix}1\\0\\0\end{bmatrix} = [T]_{SS}\,(f_0)_S = \left(T[f_0]\right)_S = \left(T[0\,f_0+0\,f_1+1\,f_2]\right)_S = \begin{bmatrix}0\\0\\1\end{bmatrix}, \qquad (40)$$

its second column must be

$$[T]_{SS}\begin{bmatrix}0\\1\\0\end{bmatrix} = [T]_{SS}\,(f_1)_S = \left(T[f_1]\right)_S = \left(T[0\,f_0+0\,f_1+0\,f_2]\right)_S = \begin{bmatrix}0\\0\\0\end{bmatrix}, \qquad (41)$$

and its last column must be

$$[T]_{SS}\begin{bmatrix}0\\0\\1\end{bmatrix} = [T]_{SS}\,(f_2)_S = \left(T[f_2]\right)_S = \left(T[1\,f_0+0\,f_1+0\,f_2]\right)_S = \begin{bmatrix}1\\0\\0\end{bmatrix}, \qquad (42)$$

that is we must have

$$[T]_{SS} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}. \tag{43}$$

The eigenvalues of $[T]_{SS}$ hence of T are roots λ of the characteristic equation

$$\begin{aligned} 0 &= \det\left(\lambda I - [T]_{SS}\right) = \det\begin{bmatrix} \lambda & 0 & -1 \\ 0 & \lambda & 0 \\ -1 & 0 & \lambda \end{bmatrix} = \lambda \det\begin{bmatrix} \lambda & -1 \\ -1 & \lambda \end{bmatrix} = \lambda\left(\lambda^2 - 1\right) \\ &= (\lambda+1)\lambda(\lambda-1), \end{aligned} \tag{44}$$

and, so, are $-1, 0, +1$. The corresponding eigenspace of $[T]_{SS}$ are then given by

$$\begin{aligned} \mathcal{E}_{-1}\left([T]_{SS}\right) &= \mathrm{Nul}\left(-1I - [T]_{SS}\right) = \mathrm{Nul}\begin{bmatrix} -1 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & -1 \end{bmatrix} = \mathrm{Nul}\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ &= \mathrm{Span}\left\{\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}\right\} = \mathrm{Span}\left\{\left(f_0 - f_2\right)_S\right\}, \\ \mathcal{E}_0\left([T]_{SS}\right) &= \mathrm{Nul}\left(0I - [T]_{SS}\right) = \mathrm{Nul}\begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} = \mathrm{Nul}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \\ &= \mathrm{Span}\left\{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right\} = \mathrm{Span}\left\{\left(f_1\right)_S\right\}, \\ \mathcal{E}_1\left([T]_{SS}\right) &= \mathrm{Nul}\left(1I - [T]_{SS}\right) = \mathrm{Nul}\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} = \mathrm{Nul}\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ &= \mathrm{Span}\left\{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right\} = \mathrm{Span}\left\{\left(f_0 + f_2\right)_S\right\}, \end{aligned} \tag{45}$$

and, so the corresponding eigenspaces of T itself are given by

$$\mathcal{E}_{-1}(T) = \mathrm{Span}\{f_0 - f_2\}, \qquad \mathcal{E}_0(T) = \mathrm{Span}\{f_1\}, \qquad \mathcal{E}_1(T) = \mathrm{Span}\{f_0 + f_2\}. \tag{46}$$

11) Suppose some linear operator $T:V\to V$ has the property that

$$T\left[f_0-f_2\right]=-1\left(f_0-f_2\right),\quad T\left[f_1\right]=0=0\,f_1,\quad T\left[f_0+f_2\right]=+1\left(f_0+f_2\right),\tag{47}$$

where $S=\left\{f_0,f_1,f_2\right\}$ is a basis for (three-dimensional) vector space V . Find a formula for

$$T^m\left[\alpha f_0+\beta f_1+\gamma f_2\right]\tag{48}$$

where α,β,γ are arbitrary scalars, where $m\in\{1,2,3,\ldots\}=\mathbb{N}$, and where T^m denotes the m^{th} composition of T with itself:

$$\begin{aligned}
&T^1\left[\alpha f_0+\beta f_1+\gamma f_2\right]:=T\left[\alpha f_0+\beta f_1+\gamma f_2\right],\\
&T^2\left[\alpha f_0+\beta f_1+\gamma f_2\right]:=T\left[T\left[\alpha f_0+\beta f_1+\gamma f_2\right]\right],\\
&T^3\left[\alpha f_0+\beta f_1+\gamma f_2\right]:=T\left[T\left[T\left[\alpha f_0+\beta f_1+\gamma f_2\right]\right]\right],\\
&T^4\left[\alpha f_0+\beta f_1+\gamma f_2\right]:=T\left[T\left[T\left[T\left[\alpha f_0+\beta f_1+\gamma f_2\right]\right]\right]\right],
\end{aligned}\tag{49}$$

etc. That is, in writing

$$T^m\left[\alpha f_0+\beta f_1+\gamma f_2\right]=C_0 f_0+C_1 f_1+C_2 f_2,\tag{50}$$

find C_0,C_1,C_2, which will in general be functions of α,β,γ as well as m . (47) will be extremely useful to compute this formula.

<u>20pts</u>

<u>Solution</u>

Make definitions

$$F_{-1}=f_0-f_2,\quad F_0=f_1,\quad F_1=f_0+f_2,\tag{51}$$

with inverse transformations evidently given by

$$f_0=\frac{1}{2}\left(F_1+F_{-1}\right),\quad f_1=F_0,\quad f_2=\frac{1}{2}\left(F_1-F_{-1}\right),\tag{52}$$

so that (47) is simply

$$T\left[F_{-1}\right]=-1F_{-1},\quad T\left[F_0\right]=0F_0,\quad T\left[F_1\right]=+1F_1,\tag{53}$$

while (48) is then

$$
\begin{aligned}
T^m\left[\alpha f_0+\beta f_1+\gamma f_2\right] &= T^m\left[\alpha\frac{1}{2}\left(F_1+F_{-1}\right)+\beta F_0+\gamma\frac{1}{2}\left(F_1-F_{-1}\right)\right]\\
&= T^m\left[\beta F_0+\frac{1}{2}(\alpha+\gamma)F_1+\frac{1}{2}(\alpha-\gamma)F_{-1}\right]\\
&= \beta T^m\left[F_0\right]+\frac{1}{2}(\alpha+\gamma)T^m\left[F_1\right]+\frac{1}{2}(\alpha-\gamma)T^m\left[F_{-1}\right]\\
&= \beta\cdot 0^m F_0+\frac{1}{2}(\alpha+\gamma)\cdot 1^m F_1+\frac{1}{2}(\alpha-\gamma)\cdot(-1)^m F_{-1}\\
&= \frac{1}{2}(\alpha+\gamma)F_1+(-1)^m\frac{1}{2}(\alpha-\gamma)F_{-1}\\
&= \frac{1}{2}(\alpha+\gamma)(f_0+f_2)+(-1)^m\frac{1}{2}(\alpha-\gamma)(f_0-f_2)\\
&= \frac{1}{2}\Big((\alpha+\gamma)+(-1)^m(\alpha-\gamma)\Big)f_0+\frac{1}{2}\Big((\alpha+\gamma)-(-1)^m(\alpha-\gamma)\Big)f_2\\
&= \left(\frac{1+(-1)^m}{2}\alpha+\frac{1-(-1)^m}{2}\gamma\right)f_0+\left(\frac{1-(-1)^m}{2}\alpha+\frac{1+(-1)^m}{2}\gamma\right)f_2, \qquad (54)
\end{aligned}
$$

where we used linearity of the composite mapping in addition to (52) and (53), etc. Thus, for $m\in\mathbb{N}$, (50) holds with

$$
C_0=\frac{1+(-1)^m}{2}\alpha+\frac{1-(-1)^m}{2}\gamma, \qquad C_1=0, \qquad C_2=\frac{1-(-1)^m}{2}\alpha+\frac{1+(-1)^m}{2}\gamma. \quad (55)
$$

12) Orthogonally diagonalize the matrix

$$
A=\begin{bmatrix}0&0&1\\0&0&0\\1&0&0\end{bmatrix}, \qquad (56)
$$

and use this to compute a formula for (each entry of) A^m, $m\in\{1,2,3,\ldots\}=\mathbb{N}$.

20pts

Solution

This begins the same as in equations (44), where one finds then that

$$
\mathcal{E}_{-1}(A)=\operatorname{Span}\left\{\begin{bmatrix}1\\0\\-1\end{bmatrix}\right\}, \qquad \mathcal{E}_0(A)=\operatorname{Span}\left\{\begin{bmatrix}0\\1\\0\end{bmatrix}\right\}, \qquad \mathcal{E}_1(A)=\operatorname{Span}\left\{\begin{bmatrix}1\\0\\1\end{bmatrix}\right\}, \qquad (57)
$$

and then that

$$A = QDQ^{-1} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}^{-1}$$

$$= PDP^{-1} = PDP^{T} = \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}. \quad (58)$$

So then for $m \in \{1, 2, 3, \ldots\} = \mathbb{N}$,

$$A^m = PD^mP^{-1} = PD^mP^T = \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} (-1)^m & 0 & 0 \\ 0 & 0^m & 0 \\ 0 & 0 & 1^m \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} (-1)^m/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 0 & 0 \\ -(-1)^m/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} (-1)^m/2 & 0 & 1/2 \\ 0 & 0 & 0 \\ -(-1)^m/2 & 0 & 1/2 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \left(1+(-1)^m\right)/2 & 0 & \left(1-(-1)^m\right)/2 \\ 0 & 0 & 0 \\ \left(1-(-1)^m\right)/2 & 0 & \left(1+(-1)^m\right)/2 \end{bmatrix}.$$

(59)