Math 313 Midterm I KEY
Spring 2010
section 001
Instructor: Scott Glasgow

Do NOT write on this problem statement booklet, except for your indication of following the honor code just below. No credit will be given for work written on this booklet. Rather write in a blue book. Also write your name, course, and section number on the blue book.

Complete only that number of questions that gives you just over 150 points—other problems you attempt after that total will not be graded.

1) What's the difference between Gauss-Jordan elimination and Gaussian elimination? Which, in general, is quicker for large systems solved by a computer? (This is as opposed to small systems solved by people by hand.)

5pts

Solution

In Gauss-Jordan elimination we row reduce the augmented matrix for a linear system of equations (or perhaps not augmented if the system is homogeneous since "zeros don't change under row reduction"—i.e. linear combinations and permutations of zeros remain zeros) until it is in reduced row-echelon form, in which case both pivot and free variables and "final" right-hand sides (zero or not) are entirely obvious. On the other hand, in Gaussian elimination we only reduce the augmented matrix until it is in row-echelon form. Here pivot variables are obvious—and the same as in Gauss-Jordan elimination—but the free variables may not yet be obvious, nor the final right hand sides. So then we must continue to (effectively) find the latter through "back-substitution", which involves solving for earlier/more-left pivot variables in terms of later/more-right variables (pivot or not). Despite the more wordy description, the latter elimination is actually faster in general—it requires only about $2/3^{\text{rd}}$'s as many operations for large (generally-structured) matrices. (So Gauss-Jordan elimination is 50% slower—i.e. it requires half again as much time—as Gaussian elimination for large general matrices).

2) ***When*** (i.e. under what circumstances) can we add matrices? ***How*** do we add matrices? When are two matrices "equal"? ***When*** can we multiply matrices? ***How*** do we multiply them? ***When*** can we multiply a scalar and a matrix? ***How*** do we multiply them? ***When*** can we take the transpose of a matrix? ***How*** do we take the transpose of a matrix? ***When*** can we take the trace of a matrix? ***How*** do we take the trace of a matrix? ***When*** can we take the determinant of a matrix? ***What*** is the definition of the determinant of a matrix? (The last question may require a paragraph, or one nice formula in which you explain the meaning of each term.)

7pts

Solution

We can add two matrices together if and only if the matrices have exactly the same size, i.e. if the two matrices have exactly the same number of rows and columns. In such case we simply add corresponding entries. Two matrices are “equal” iff they are the same size and the corresponding entries are all equal. We can multiply two matrices iff the matrix on the left of the product has as many columns as the matrix on the right of the product has rows. The formula for multiplying an $m \times r$ matrix A into (i.e. from the left into) an $r \times n$ matrix B is

$$\left(AB\right)_{ij} = \sum_{k=1}^{r} A_{ik} B_{kj}. \tag{1}$$

We can always multiply a scalar into a matrix. We do so by multiplying the scalar into each entry of the matrix. We can always take the transpose of a matrix, and do so by exchanging rows for columns, in order. We only take the trace of square matrices, in which case we add up the entries on the main diagonal. We only take the determinant of square matrices. The definition of determinant is the sum of all of the signed elementary products of the square matrix. An elementary product is a product of entries from a matrix in which one and only one (linear) factor is taken from each row and column. The sign of this product is determined by the number of, say, “column inversions” necessary to produce this combination of factors by permuting the column labels from the “main diagonal” elementary product; the sign is plus if the number of inversions is even, minus otherwise. In this setting, inversions are defined by a column label of one of these factors being bigger than a column label of a subsequent factor, the row labels of factors proceeding in order from smallest to largest (as factors are written from left to right).

3) True or False: the product $A^T A$ is always well-defined (for any size matrix A). Justify you answer.

5pts

Solution

True: the number of columns of the matrix on the left will always be the same as the number of rows of the matrix on the right.

4) True or False: The product of a singular matrix and an invertible matrix (of the same size) may sometimes by invertible. Justify your answer by stating a theorem or giving a (positive) example of the claimed phenomenon.

7pts

Solution

False. The theorem is that the product of (square) matrices (of the same size) is invertible iff each matrix factor is invertible. To see the relevant direction we need for this particular question (which the student is not required to produce), suppose B is singular in the product AB. Then the equation

$$B\mathbf{x} = \mathbf{0} \tag{2}$$

has more than just the trivial solution $\mathbf{x} = \mathbf{0}$. But for such a nonzero solution $\mathbf{x}$ of (2) we also get

$$\mathbf{0} = A\mathbf{0} = A\left(B\mathbf{x}\right) = \left(AB\right)\mathbf{x}, \tag{3}$$

i.e. the equation $\left(AB\right)\mathbf{x} = \mathbf{0}$ has a nontrivial solution $\mathbf{x}$ and, so, by equivalent statements, AB is singular. (Here we used the associative property of matrix multiplication.) Similarly, suppose A is singular in the product AB. Then by previous theorem we know that A^T is singular (by that theorem, if A^T were invertible, then so would be $\left(A^T\right)^T = A$). Thus the equation

$$A^T\mathbf{x} = \mathbf{0} \tag{4}$$

must have a nontrivial solution, as also then the equation

$$\left(B^T A^T\right)\mathbf{x} = \mathbf{0} \tag{5}$$

(where we again used the associative property), so that by equivalent statements $B^T A^T = \left(AB\right)^T$ is singular, hence AB itself is singular.

5) Put the following matrix in reduced row-echelon form

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 4 & -2 & 1 & 4 \end{bmatrix} \tag{6}$$

<u>10pts</u>

<u>Solution</u>

The row reduction may proceed as follows:

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 4 & -2 & 1 & 4 \end{bmatrix} \overset{\substack{R2-R1 \\ R3-4R1}}{\sim} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & 0 & -2 \\ 0 & -6 & -3 & 0 \end{bmatrix} \overset{\substack{R2/(-2) \\ R3/(-3)}}{\sim} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 2 & 1 & 0 \end{bmatrix} \overset{R3-2R2}{\sim} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix} \overset{R1-R3}{\sim} \begin{bmatrix} 1 & 1 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix} \overset{R1-R2}{\sim} \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix}. \tag{7}$$

However the row reduction proceeds, the row echelon form is unique—the last matrix indicated in (7) is ***the*** answer.

6) Solve the following system of 2 equations and 2 unknowns by performing Gauss-Jordon elimination on the relevant augmented matrix.

$$\begin{aligned} x + y &= 3, \\ x + 2y &= 5. \end{aligned} \tag{8}$$

<u>7pts</u>

<u>Solution</u>

The augmented matrix and its row reduction appear below:

$$\begin{bmatrix} 1 & 1 & 3 \\ 1 & 2 & 5 \end{bmatrix} \overset{R_2-R_1}{\sim} \begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & 2 \end{bmatrix} \overset{R_1-R_2}{\sim} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}. \tag{9}$$

Thus the solution to (8) is $(x, y) = (1, 2)$.

7) Under what circumstances can a square matrix be written as a product of a finite number of elementary matrices?

<u>5pts</u>

<u>Solution</u>

By equivalent statements, this representation occurs iff the matrix is invertible, and, so, iff any one of the other equivalent statements.

8) Write the matrix

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \tag{10}$$

as a product of 2 elementary matrices.

7pts

Solution

In a previous problem we noted that the matrix is reduced to the identity matrix by 2 elementary row operations. Applying the inverses of these row operations to the relevant version of the identity, and forming the product of such in the opposite order in which they were applied to the matrix, we get the desired product: since from (9) we have

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \overset{R_2 - R_1}{\sim} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \overset{R_1 - R_2}{\sim} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I, \tag{11}$$

i.e., in what we hope is compelling notation,

$$\left(R_1 - R_2\right)\left(\left(R_2 - R_1\right)\left(A\right)\right) = I\,, \tag{12}$$

then clearly

$$\begin{aligned} A &= \left(R_2 - R_1\right)^{-1}\left(\left(R_1 - R_2\right)^{-1}\left(I\right)\right) \overset{\text{theorem}}{=} \left(R_2 - R_1\right)^{-1}\left(I\right)\left(R_1 - R_2\right)^{-1}\left(I\right) \\ &= \left(R_2 + R_1\right)\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\left(R_1 + R_2\right)\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}. \end{aligned} \tag{13}$$

Here the indicated theorem may itself not be clear—it requires a proof with 3 parts to it—but the rest should be clear. One (partially) checks (this theorem) by noting that in fact

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} = A\,. \tag{14}$$

9) Find the inverse of the matrix A of the previous problem by either row-reducing a certain relevant augmented matrix, or by using the results of the previous problem itself in an insightful way. Do not use “the formula”, i.e. do not use the adjoint formula for an inverse.

8pts

Solution

The augmented matrix to be reduced is $\left[A \middle| I\right]$, which reduces to $\left[I \middle| A^{-1}\right]$ when the matrix has an inverse—the matrix in this problem certainly does. The actual proof of this fact is more along the lines of "using the results of the previous problem in an insightful way": from that problem and relevant theorems we certainly have

$$\begin{aligned} A^{-1} &= \left(\left(R_2 - R_1\right)^{-1}(I)\left(R_1 - R_2\right)^{-1}(I)\right)^{-1} = \left(\left(R_1 - R_2\right)^{-1}(I)\right)^{-1}\left(\left(R_2 - R_1\right)^{-1}(I)\right)^{-1} \\ &= \left(R_1 - R_2\right)\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\left(R_2 - R_1\right)\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}. \end{aligned} \tag{15}$$

Either way one checks that

$$\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}. \tag{16}$$

10) Find all $\mathbf{b}$'s such that $A\mathbf{x} = \mathbf{b}$ has a solution $\mathbf{x}$, where A is the singular matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}. \tag{17}$$

You may do this in the "straightforward way", or you might consider using the following (as yet unproved) theorem, which is usually easier: $A\mathbf{x} = \mathbf{b}$ has a solution $\mathbf{x}$ iff $\mathbf{v}^T\mathbf{b} = 0$ (here 0 is actually the 1×1 matrix $\left[0\right]$) for every $\mathbf{v}$ satisfying $A^T\mathbf{v} = \mathbf{0}$.

8pts

Solution

Solutions $\mathbf{v}$ of $A^T\mathbf{v} = \mathbf{0}$ are gotten by the following row reduction:

$$\left[A^T\right] = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix} \overset{\substack{R_2-2R_1 \\ R_3-3R_1}}{\sim} \begin{bmatrix} 1 & 4 & 7 \\ 0 & -3 & -6 \\ 0 & -6 & -12 \end{bmatrix} \overset{\substack{R_3-2R_2 \\ R_2/-3}}{\sim} \begin{bmatrix} 1 & 4 & 7 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \overset{R_1-4R_2}{\sim} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}. \tag{18}$$

So all such $\mathbf{v}$'s are of the form

$$\mathbf{v} = t\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}. \tag{19}$$

Thus the indicated requirement is that for a right-hand side of the form $\mathbf{b}^T = (b_1, b_2, b_3)$ we must have (and this is sufficient)

$$[0] = \mathbf{v}^T\mathbf{b} = t\begin{bmatrix} 1 & -2 & 1 \end{bmatrix}\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = t\left[b_1 - 2b_2 + b_3\right] \tag{20}$$

for every t. Thus the necessary and sufficient condition is that the right-hand side $\mathbf{b}$ is of the form

$$\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ -b_1 + 2b_2 \end{bmatrix} = b_1\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + b_2\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \tag{21}$$

where b_1 and b_2 are arbitrary.

11) Find the determinant of

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 10 \end{bmatrix} \tag{22}$$

by cofactor expansion off of the last column at every step, including the second step (and without using row reduction).

8pts

Solution

$$
\begin{aligned}
\det\begin{bmatrix}1&2&3\\4&5&6\\7&8&10\end{bmatrix} &= +3\det\begin{bmatrix}4&5\\7&8\end{bmatrix}-6\det\begin{bmatrix}1&2\\7&8\end{bmatrix}+10\det\begin{bmatrix}1&2\\4&5\end{bmatrix}\\
&=3\left(-5\det[7]+8\det[4]\right)-6\left(-2\det[7]+8\det[1]\right)\\
&\quad+10\left(-2\det[4]+5\det[1]\right)\\
&=3(-35+32)-6(-14+8)+10(-8+5)=3(-3)-6(-6)\\
&\quad+10(-3)\\
&=-9+36-30=-3.
\end{aligned}
\tag{23}
$$

12) Fill in the blank for square matrix A:

$$A\,\mathrm{adj}(A)= \quad . \tag{24}$$

5pts

Solution

$$A\,\mathrm{adj}(A)=\det(A)I\ . \tag{25}$$

13) Show that if a system of linear equations

$$A\mathbf{x}=\mathbf{b} \tag{26}$$

has 2 (distinct) solutions $\mathbf{x}$ then it also has infinitely many (distinct) solutions. (Do not assume the theorem that a linear system has only 0, 1, or infinitely many solutions. Rather use matrix algebra to show this must be the case.)

10pts

Solution

If $\mathbf{x}_1$ and $\mathbf{x}_2 \neq \mathbf{x}_1$ both solve (26), then for each $t \in \mathbb{R}$, ***of which there are infinitely many***, we get (using matrix algebra) that a simple interpolation "between" these solutions gives

$$A\left((1-t)\mathbf{x}_1+t\mathbf{x}_2\right)=(1-t)A\mathbf{x}_1+tA\mathbf{x}_2=(1-t)\mathbf{b}+t\mathbf{b}=\left((1-t)+t\right)\mathbf{b}=1\mathbf{b}=\mathbf{b}\ , \tag{27}$$

i.e. there are infinitely many distinct solution solutions of (26) (of the form $\mathbf{x}=(1-t)\mathbf{x}_1+t\mathbf{x}_2=\mathbf{x}_1+t\left(\mathbf{x}_2-\mathbf{x}_1\right)$). Note that it is important that $\mathbf{x}_2 \neq \mathbf{x}_1$ to get this result as otherwise $(1-t)\mathbf{x}_1 + t\mathbf{x}_2 = ((1-t)+t)\mathbf{x}_1 = 1\mathbf{x}_1 = \mathbf{x}_1$ only represents 1 solution, not infinitely many.

14)

a) List each disordered pair in the following permutations of $(1,2,3,4,5)$:

i) $(5,4,3,2,1)$

ii) $(2,4,3,1,5)$.

7pts

b) For each of the two permutations above find the associated ***signed*** elementary product (used in the construction of the determinant) of the matrix

$$A = \begin{bmatrix} a & b & c & d & e \\ f & g & h & i & j \\ k & l & m & n & o \\ p & q & r & s & t \\ u & v & w & x & y \end{bmatrix}.$$

8pts

Solution

a) For i) there are 10 disordered pairs, which are $(5,4)$, $(5,3)$, $(5,2)$, $(5,1)$, $(4,3)$, $(4,2)$, $(4,1)$, $(3,2)$, $(3,1)$, and $(2,1)$. For ii) there are 4 disordered pairs, which are $(2,1)$, $(4,3)$, $(4,1)$, and $(3,1)$.

b) The signed elementary products are then $(-1)^{10} A_{15}A_{24}A_{33}A_{42}A_{51} = +1eimqu = eimqu$, and $(-1)^4 A_{12}A_{24}A_{33}A_{41}A_{55} = +1bimpy = bimpy$.

15) True or False: Every homogeneous system of equations

$$A\mathbf{x} = \mathbf{0} \tag{28}$$

with more unknowns than equations (hence A is a $m \times n$ matrix with $n > m$, $\mathbf{x}$ is a $n \times 1$ column matrix of unknowns, and $\mathbf{0}$ is a $m \times 1$ column matrix of zeroes), has a non-zero/nontrivial solution $\mathbf{x}$. Justify your answer for the last 5 points of this problem.

2+5=7 points

Solution

This is a true statement. The proof (which is not required here) is effectively that the row echelon form of A, which arises in row reducing the (trivially) augmented matrix $\left[A|\mathbf{0}\right]$ (hence arises in finding any and all solutions of (28)), cannot have any more pivots/leading 1's than either the number of rows or columns of A (since each pivot requires both a unique column and row to put it in). Hence this number of pivots is no more than the minimum of n and m, which in this case is m. Thus the solution $\mathbf{x}=\mathbf{x}_{n\times 1}$ has no more than $m<n$ leading variables, hence has at least $n-m\geq 1$ free variables. Even one free variable represents/gives rise to an infinite number of distinct solutions (not all of which can be $\mathbf{x}=\mathbf{0}_{n\times 1}$ then).

16) True or False: Every homogeneous system of equations

$$A\mathbf{x}=\mathbf{0} \tag{29}$$

with more unknowns than equations has infinitely many solutions. Justify your answer for the last 3 points of this problem.

1+3=4 points

Solution

The statement is true, for the same reasons indicated in/for the previous problem.

17) Suppose the system of equations

$$A\mathbf{x}=\mathbf{b} \tag{30}$$

has one and only one solution $\mathbf{x}\in\mathbb{R}^n$ for each and every $\mathbf{b}\in\mathbb{R}^n$. (Evidently A is an $n\times n$ matrix.) Now tell me whether the following statement is true or is false: "It is possible that the matrix A allows there to be an $\mathbf{x}\neq\mathbf{0}$ such that $A\mathbf{x}=\mathbf{0}$", i.e. "It is possible that the homogeneous version of equation (30) has a nontrivial solution". ***Prove your assertion***.

5 points

Solution

Without recourse to any “fancy theorems”, we have the following: $A\mathbf{x} = \mathbf{0}$ for $\mathbf{x} = \mathbf{0}$ certainly, and the hypotheses say that this is the only one. So the statement is false.

18) Suppose the system of equations

$$A\mathbf{x} = \mathbf{b} \tag{31}$$

has a solution $\mathbf{x} \in \mathbb{R}^n$ for each and every $\mathbf{b} \in \mathbb{R}^n$. (Evidently A is an $n \times n$ matrix.) Tell me whether the following statement is true or is false: “It is possible that the matrix A allows there to be an $\mathbf{x} \neq \mathbf{0}$ such that $A\mathbf{x} = \mathbf{0}$”, i.e. “It is possible that the homogeneous version of equation (31) has a nontrivial solution”. ***Prove your assertion***.

10 points

Solution

With recourse to our “Equivalent Statements”, we have that $A\mathbf{x} = \mathbf{0}$ has only the trivial solution $\mathbf{x} = \mathbf{0}$ if and only if (31) has a solution $\mathbf{x} \in \mathbb{R}^n$ for every $\mathbf{b} \in \mathbb{R}^n$. So the statement is false.

Extra Pedagogy: Recall that the proof of this equivalence goes something like this: If (31) is consistent for every choice of $\mathbf{b} \in \mathbb{R}^n$, then we can solve systems $A\mathbf{x}_i = \mathbf{b}_i$, $i = 1, \dots, n$, with the $\mathbf{b}_i$ ’s being the relevant columns of the identity matrix I. Then the matrix $C = \left[\mathbf{x}_1 | \dots | \mathbf{x}_i | \dots | \mathbf{x}_n\right]$ certainly turns out to be a “right inverse” of A, which, by theorem 1.6.3, will also be “***the*** inverse” of A, so that $A\mathbf{x} = \mathbf{0}$ has only the solution $\mathbf{x} = I\mathbf{x} = \left(A^{-1}A\right)\mathbf{x} = A^{-1}\left(A\mathbf{x}\right) = A^{-1}\mathbf{0} = \mathbf{0}$. [Note that theorem 1.6.3 could itself be proved there by showing that this right inverse is itself invertible: consider the system $C\mathbf{x} = \mathbf{0}$, which then has only the solution $\mathbf{x} = I\mathbf{x} = \left(AC\right)\mathbf{x} = A\left(C\mathbf{x}\right) = A\mathbf{0} = \mathbf{0}$, and which, according to equivalent statements, gives C invertible. Then using that, under this circumstance, $AC = I \Rightarrow A = C^{-1} \Rightarrow CA = CC^{-1} = I$, which says (among other things) that C is also A ’s inverse.] This is enough of the equivalence to deduce that the statement is false.

To get the other direction in this equivalence, and to illuminate how some of the other equivalent statements just used are indeed equivalent, recall the following: if $A\mathbf{x} = \mathbf{0}$ has only the trivial solution $\mathbf{x} = \mathbf{0}$, then row reduction of $\left[A | \mathbf{0}\right]$ must give $\left[I | \mathbf{0}\right]$ *in a finite number of steps*. (This presumes that row reduction does not change the solution space, and that we, as indicated, “get done” in a finite number of steps. We have never really proved this!) So (in finite number of steps) row reduction of A must give I (ignoring the last columns of zeroes in the above augmented matrices), which shows (with theorem 1.5.1) that the product of A with a finite number of elementary matrices is I, which then shows that A can be expressed as a product of (the inverses of these) elementary matrices, and, so, is itself invertible, giving (finally!) that $A\left(A^{-1}\mathbf{b}\right)=\left(AA^{-1}\right)\mathbf{b}=I\mathbf{b}=\mathbf{b}$ for every $\mathbf{b}\in\mathbb{R}^n$, so that (31) has at least the solution $\mathbf{x}=A^{-1}\mathbf{b}$ for every $\mathbf{b}$. Aside from row reduction preserving the solution space of a system of equations (and getting done in a finite number of steps), the other "big idea" that may be buried in here is the fact that elementary matrices, or, more to the point, elementary row operations, are "truly" invertible, i.e. that the left or right inverses of such are in fact also right and left inverses. This last statement formed in terms of elementary row operations is the following: not only is it the case that for every elementary row operation there is another one that will "undo" it "afterwards", but that same "afterwards inverse" done "before" the given elementary row operation will itself be undone by the given elementary row operation. Of course this distinction of "before" and "after" is at the heart of what we mean by "right" and "left" inverses!

Note then that at one very basic level, the truth of all of our equivalent statements comes down to row operations, specifically that a) they don't alter the solution space of a system of equations, that b) only a finite number of them is required to put a matrix in (even) (reduced) row echelon form, and that c) they are all "before/after"= "left\right" invertible. Perhaps these 3 claims should be thoroughly investigated by the serious student.

19) Suppose the system of equations

$$A\mathbf{x}=\mathbf{b} \tag{32}$$

has no solution $\mathbf{x}\in\mathbb{R}^n$ for some particular $\mathbf{b}\in\mathbb{R}^n$. (Here A is an $n\times n$ matrix.) Tell me whether the following statement is true or is false: "There is another right-hand-side $\mathbf{b}\in\mathbb{R}^n$ such that (32) has infinitely many solutions." ***Prove your assertion***.

15 points

<u>Solution</u>

By (the contrapositive/negation of our stated) equivalent statements the supposition gives us that, for example, $A\mathbf{x}=\mathbf{0}$ has more than just one solution. (Here we have chosen the "other $\mathbf{b}$" to be $\mathbf{0}$. Mind you $\mathbf{0}$ really is *another* $\mathbf{b}$ since for $\mathbf{b}=\mathbf{0}$ (32) actually has a solution, namely $\mathbf{x}=\mathbf{0}$, contrary to the supposition.) But since the possibilities for the number of solutions of linear systems is only 0, 1, or ∞, there must be an infinite number of solutions: the statement is true.

20) Suppose the system of equations

$$A\mathbf{x}=\mathbf{b} \tag{33}$$

(33) has no solution $\mathbf{x}\in\mathbb{R}^n$ for some particular $\mathbf{b}\in\mathbb{R}^n$. (Here A is an $n\times n$ matrix.) Suppose also that A is not the zero matrix, i.e. not all of its entries are zero. Tell me

whether the following statement is true or is false: “There is another ***nonzero*** right-hand-side $\mathbf{b} \in \mathbb{R}^n$ such that (32) has infinitely many solutions.” ***Prove your assertion***.

20 points

Solution

By equivalent statements the supposition gives that there is a nonzero solution to $A\mathbf{x} = \mathbf{0}$, call it $\mathbf{x}_0$ ($\neq \mathbf{0}$). Now pick an $\mathbf{x} \in \mathbb{R}^n$, call it $\mathbf{x}_1$, with zeroes in every entry/row except for the number 1placed in the row corresponding to a (favorite) nonzero column $\mathbf{c}$ of A. Then by the rules of matrix multiplication, $A\mathbf{x}_1 = \mathbf{c}$ and for each $t \in \mathbb{R}$, ***of which there are infinitely many***, we have

$$A\left(\mathbf{x}_1 + t\mathbf{x}_0\right) = A\mathbf{x}_1 + tA\mathbf{x}_0 = \mathbf{c} + t\mathbf{0} = \mathbf{c} \neq \mathbf{0}. \tag{34}$$

So we see there is in fact a nonzero right-hand-side $\mathbf{b} \in \mathbb{R}^n$—namely the nonzero column $\mathbf{c}$ of A—such that (33) has infinitely many (distinct) solutions: the statement is true. (Note it is important that $\mathbf{x}_0 \neq \mathbf{0}$ as otherwise $\mathbf{x}_1 + t\mathbf{x}_0$ represents only 1 solution, not infinitely many, as t ranges over the reals.)

21) Suppose the system of equations

$$A\mathbf{x} = \mathbf{b} \tag{35}$$

has infinitely many solutions $\mathbf{x} \in \mathbb{R}^n$ for some particular $\mathbf{b} \in \mathbb{R}^n$. (Evidently A is an $n \times n$ matrix.) Tell me whether the following statement is true or is false: “There is another right-hand-side $\mathbf{b} \in \mathbb{R}^n$ such that (35) has no solutions.” ***Prove your assertion***.

15 points

Solution

By equivalent statements the supposition gives us that there is a $\mathbf{b} \in \mathbb{R}^n$ for which (35) has no solution: the statement is true.

22) Assuming A and B are invertible matrices of the same size, show that

$$\left(AB\right)^{-1} = B^{-1}A^{-1}. \tag{36}$$

15 points

Solution

$B^{-1}A^{-1}$ is the inverse of AB if and only if

$$(AB)\left(B^{-1}A^{-1}\right)=\left(B^{-1}A^{-1}\right)(AB)=I, \tag{37}$$

to whit we first note that, by the associative property of matrix multiplication,

$$\begin{gathered}(AB)\left(B^{-1}A^{-1}\right)=\left(A\left(BB^{-1}\right)\right)A^{-1}\\ \text{and}\\ \left(B^{-1}A^{-1}\right)(AB)=\left(B^{-1}\left(A^{-1}A\right)\right)B.\end{gathered} \tag{38}$$

Then, by the definition of the inverses, in particular that an inverse is both a right and a left inverse, we have, respectively, that

$$\begin{gathered}(AB)\left(B^{-1}A^{-1}\right)=\left(A\left(BB^{-1}\right)\right)A^{-1}=(AI)A^{-1}\\ \text{and}\\ \left(B^{-1}A^{-1}\right)(AB)=\left(B^{-1}\left(A^{-1}A\right)\right)B=\left(B^{-1}I\right)B.\end{gathered} \tag{39}$$

Using now the fact that the identity matrix is in fact the "multiplicative identity" *from either side* we get

$$\begin{gathered}(AB)\left(B^{-1}A^{-1}\right)=(AI)A^{-1}=AA^{-1}\\ \text{and}\\ \left(B^{-1}A^{-1}\right)(AB)=\left(B^{-1}I\right)B=B^{-1}B.\end{gathered} \tag{40}$$

Finally we use again the definition of the inverses. In particular, using that an inverse is both a right and a left inverse, we have, respectively, that

$$\begin{gathered}(AB)\left(B^{-1}A^{-1}\right)=AA^{-1}=I\\ \text{and}\\ \left(B^{-1}A^{-1}\right)(AB)=B^{-1}B=I,\end{gathered} \tag{41}$$

which is the required (37)

23) By using the (permutation) definition of the determinant, prove that if square matrix B is the same as square matrix A, except that one of B's rows is a scalar multiple k of the corresponding row of A, then

$$\det B = k \det A. \tag{42}$$

10 points

Solution

Let it be the j^{th} row of the matrix B that is the same as the scalar k multiplied by the j^{th} row of the matrix A. Then, by definition of the determinant, and the relationship between the 2 rows (of matrix A and matrix B), we have

$$\begin{aligned}\det B &:= \sum_{p:[n]\to[n]} (-1)^{d(p)} b_{1p(1)} \cdots b_{jp(j)} \cdots b_{np(n)} = \sum_{p:[n]\to[n]} (-1)^{d(p)} a_{1p(1)} \cdots \left(k a_{jp(j)}\right) \cdots a_{np(n)} \\ &= k \sum_{p:[n]\to[n]} (-1)^{d(p)} a_{1p(1)} \cdots a_{jp(j)} \cdots a_{np(n)} =: k \det A.\end{aligned} \tag{43}$$

24) Show that if a matrix A is invertible, the system $A\mathbf{x} = \mathbf{b}$ has one and only one solution $\mathbf{x}$, namely $\mathbf{x} = A^{-1}\mathbf{b}$. [Warning: there are two things to prove here, namely a) that ***if*** the system has a solution, then it can only be $\mathbf{x} = A^{-1}\mathbf{b}$, and that b) $\mathbf{x} = A^{-1}\mathbf{b}$ actually does solve the system. Here then you will have addressed the "one and only one" issues in reverse order: you may first show that a) there is at most one solution, and b) that there is in fact one solution (rather than none). In parts a) and b) you will use that A^{-1} is A's left and right inverse, ***respectively***.]

15 points

Solution

If the system has a solution $\mathbf{x}$, then, for any such $\mathbf{x}$, we may certainly write

$$A\mathbf{x} = \mathbf{b} \tag{44}$$

without implicitly lying, and then, by left application of A^{-1} to (44), as well as by the associative property of matrix multiplication, obtain that

$$\mathbf{x} = I\mathbf{x} = \left(A^{-1}A\right)\mathbf{x} = A^{-1}\left(A\mathbf{x}\right) = A^{-1}\mathbf{b}\,. \tag{45}$$

In (45) we also used that A^{-1} is a left inverse of A, as well as the fact that the so-called identity matrix I is in fact a "multiplicative identity". Here then we have just showed that if (44) has a solution, it's got to be $\mathbf{x} = A^{-1}\mathbf{b}$. Thus we have showed that (44) has at most one solution. But our demonstration does not yet preclude there being no solution. To preclude that possibility, we confirm that, for the only promising candidate, namely $\mathbf{x} = A^{-1}\mathbf{b}$, we get

$$A\mathbf{x} = A\left(A^{-1}\mathbf{b}\right) = \left(AA^{-1}\right)\mathbf{b} = I\mathbf{b} = \mathbf{b}, \tag{46}$$

so that our candidate was successful. (Here we have used the associative property of matrix multiplication, the fact that A^{-1} is a right inverse of A, as well as the fact that the so-called identity matrix I is in fact a "multiplicative identity".) Thus we have showed that the system has one and only one solution, namely $\mathbf{x} = A^{-1}\mathbf{b}$.

25) Suppose B is a left inverse of square matrix A, i.e. suppose

$$BA = I. \tag{47}$$

(And, so, I and B are the same size as A.) By considering solutions $\mathbf{x}$ of the system

$$A\mathbf{x} = \mathbf{0}, \tag{48}$$

and by use of equivalent statements (and use of the hypothesis (47) of course!), show that A must have an inverse A^{-1}, i.e. it must have a left-right inverse (and, so, by essentially the results of the next problem, $B = A^{-1}$, or, equivalently, (47) ultimately implies $AB = I$.)

15 points

Solution

With hypothesis (47), one finds the one and only solution $\mathbf{x}$ of (48) is $\mathbf{x} = \mathbf{0}$: $\mathbf{x} = \mathbf{0}$ certainly satisfies (48) and whenever (48) has a solution $\mathbf{x}$ one finds

$$\mathbf{x} = I\mathbf{x} = \left(BA\right)\mathbf{x} = B\left(A\mathbf{x}\right) = B\mathbf{0} = \mathbf{0}. \tag{49}$$

So by equivalent statements, A is invertible, which means it has a left-right inverse. (Thus $B = A^{-1}$:

$$B = BI = B\left(AA^{-1}\right) = \left(BA\right)A^{-1} = IA^{-1} = A^{-1}. \tag{50}$$

This is nearly the result of the next problem.)

26) Assume that both the matrix B and the matrix C are inverses of the matrix A. Show that B and C are just two aliases for the same matrix, i.e. show that in fact $B = C$.

10 points

Solution

The descriptions of B and C demand that

$$AB = BA = I = AC = CA. \tag{51}$$

Using the associative property of matrix multiplication in two different ways on the product BAC we get

$$\begin{gathered} BAC = B(AC) = BI = B \\ \text{and} \\ BAC = (BA)C = IC = C, \end{gathered} \tag{52}$$

so that indeed

$$\begin{gathered} B = BAC = C \\ \Rightarrow \\ B = C \end{gathered} \tag{53}$$

as claimed. Note that in (52) we also used that a) C is a right inverse of A, b) B is a left inverse of A, and that c) the identity matrix acts as both a right and left multiplicative identity. One can alternately approach this problem by considering the product CAB, but then by using that a) C is a left inverse of A, b) B is a right inverse of A, and again that c) the identity matrix acts as both a right and left multiplicative identity.

27) Find the inverse of the matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \tag{54}$$

by row reducing $\left[A \middle| I \right]$ to $\left[I \middle| A^{-1} \right]$. Assume the parameters $a, b, c,$ and d do not take on any special values, nor have a special relationship among them—that is row reduce naively, without worrying about any divisions by “hidden zeros”.

10 points

Solution

The naïve row reduction mentioned may proceed as follows:

$$
\begin{aligned}
\left[A\middle|I\right] &= \left[\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right]^{aR2-cR1} \sim \left[\begin{array}{cc|cc} a & b & 1 & 0 \\ 0 & ad-bc & -c & a \end{array}\right]^{(ad-bc)R1-bR2} \\
&\sim \left[\begin{array}{cc|cc} a(ad-bc) & 0 & ad & -ab \\ 0 & ad-bc & -c & a \end{array}\right]^{R1/a} \sim \left[\begin{array}{cc|cc} ad-bc & 0 & d & -b \\ 0 & ad-bc & -c & a \end{array}\right]^{R1/a} \\
&\sim \left[\begin{array}{cc|c} 1 & 0 & \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \\ 0 & 1 & \end{array}\right]^{R1/a} = \left[I\middle|A^{-1}\right].
\end{aligned}
\tag{55}
$$

28) When is it (precisely) that the product of two symmetric matrices is symmetric? Prove this. (Assume of course that the product of the two matrices makes sense, i.e. assume the two matrices are the same size.)

10 points

<u>Solution</u>

The product AB of two matrices A and B is, by definition, “symmetric” if and only if $AB = (AB)^T$. On the other hand we have proved that $(AB)^T = B^T A^T$, which, under the present circumstances, is the product BA. Thus, if the matrices A and B are symmetric, their product is symmetric if and only if $AB = BA$, i.e. if and only if A and B commute.