Math 313 Midterm II KEY
Spring 2010
section 001
Instructor: Scott Glasgow

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Signature:

1) Find the standard matrix for the following linear operator on $\mathbb{R}^3$: A rotation of 180° counter clockwise about the z axis, followed by a rotation of 90° counter clockwise about the y axis, followed by a rotation of 270° counter clockwise about the x axis.

15pts

Solution

By theorem we have that for this linear operator $T:\mathbb{R}^3 \to \mathbb{R}^3$,

$$[T] = \left[T(\hat{x})\middle|T(\hat{y})\middle|T(\hat{z})\right] = \left[T\begin{pmatrix}1\\0\\0\end{pmatrix}\middle|T\begin{pmatrix}0\\1\\0\end{pmatrix}\middle|T\begin{pmatrix}0\\0\\1\end{pmatrix}\right]. \tag{1}$$

Now $\hat{x} = (1,0,0)^T$ is sent to $-\hat{x} = (-1,0,0)^T$ by the 180° counter clockwise rotation about the z axis, and the rotation of 90° counter clockwise about the y axis sends it to $\hat{z} = (0,0,1)^T$, and then the rotation of 270° counter clockwise about the x axis sends this to $\hat{y} = (0,1,0)^T$. Similarly, $\hat{y} = (0,1,0)^T$ is changed to $-\hat{y} = (0,-1,0)^T$ by the 180° counter clockwise rotation about the z axis, and the latter is unchanged by a rotation about the y axis, which then goes to $\hat{z} = (0,0,1)^T$ via a rotation of 270° counter clockwise about the x axis. Finally $\hat{z} = (0,0,1)^T$ is unchanged by a rotation about its axis, which then is changed to $\hat{x} = (1,0,0)^T$ by a rotation of 90° counter clockwise about the y axis, which then is fixed by a rotation by that axis. Thus,

$$[T] = \left[T(\hat{x})\middle|T(\hat{y})\middle|T(\hat{z})\right] = \left[\hat{y}\middle|\hat{z}\middle|\hat{x}\right] = \begin{bmatrix}0&0&1\\1&0&0\\0&1&0\end{bmatrix}. \tag{2}$$

If these operations are composed in the opposite order, one would get the matrix

$$\left[-\hat{z}\middle|\hat{x}\middle|-\hat{y}\right] = \begin{bmatrix}0&1&0\\0&0&-1\\-1&0&0\end{bmatrix},$$

which is incorrect.

2) Determine whether multiplication by matrix A is one-to-one:

a)

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad (3)$$

b)

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}. \quad (4)$$

15pts

Solution

By theorem, multiplication by ($m \times n$) matrix A—i.e. "T_A"—is one-to-one iff the only solution $\mathbf{x}$ to $A\mathbf{x} = \mathbf{0}$ is the solution $\mathbf{x} = \mathbf{0}$, which we can find by row reduction. But since these matrices are already in reduced row echelon form, the (nature of) the solution space of $A\mathbf{x} = \mathbf{0}$ is already clear: the first system has only the solution $\mathbf{x} = (x_1, x_2)^T = \mathbf{0} \in \mathbb{R}^2$, while the second has solutions of the form $\mathbf{x} = (x_1, x_2, x_3)^T = (-s, -s, s)^T = -s(1,1,-1)^T \in \mathbb{R}^3$, not all of which being zero. So the first is one-to-one, the second many-to-one.

3) Indicate whether each of the following statements is (always) true or sometimes true or always false. Justify your answer by theorem, definition or counterexample.
a) If T maps $\mathbb{R}^n$ into $\mathbb{R}^m$ and is linear, and if $n < m$, then T is one-to-one.
b) If T maps $\mathbb{R}^n$ into $\mathbb{R}^m$ and is linear, and if $n > m$, then T is one-to-one.
c) If T maps $\mathbb{R}^n$ into $\mathbb{R}^m$ and is linear, and if $m = n$, then T is one-to-one.

15pts

Solution

a) This is "usually true", i.e. "sometimes true". An example is multiplication by matrix A of the first part of problem1, giving a map from $\mathbb{R}^2$ into $\mathbb{R}^3$. It is not always true: take for (an extreme) example the same size matrix of the first part of problem 1 but with all zero entries.
b) This is never true, i.e. always false: Said T will have a matrix $[T]$ which has more columns than rows, so that the homogeneous system $[T]\mathbf{x} = \mathbf{0}$ will have more unknowns than equation, which, theorem, always gives an infinitude of nontrivial solutions (since the row reduced form will have no more than $m < n$ pivot variables, leaving at least $n - m \geq 1$ free variables). Thus the pre-images $\mathbf{x}$ of image $\mathbf{0}$ are infinitely many, and the map is clearly many-to-one, "at least for image $\mathbf{0}$." (By linearity, more can be said.)

c) This is "usually true", i.e. "sometimes true". By equivalent statements, these examples are precisely those of multiplication by an invertible square matrix A. Since not all square matrices are invertible, the statement is not always true: take for an extreme example a square matrix with all zero entries.

4) For the given set of objects, together with the indicated notions of addition and scalar multiplication, determine whether each of the ten vector space axioms holds: real pairs (x, y), where

$$(x,y)+(x',y') := (x+x', y+y'), \quad k(x,y) := \left(\frac{kx+2ky}{5}, \frac{2kx+4ky}{5}\right). \tag{5}$$

20pts

Solution

1) through 5) : Since $V = \mathbb{R}^2$ but with only scalar multiplication differing, these axioms hold (since they reference only vector addition).

6) $k(x,y) \in V$ when $(x,y) \in V$ and $k \in \mathbb{R}$ since both $\frac{kx+2ky}{5}$ and $\frac{2kx+4ky}{5}$ are clearly real numbers then.

7) We have

$$\begin{aligned} k\big((x,y)+(w,z)\big) = k(x+w, y+z) &= \left(\frac{k(x+w)+2k(y+z)}{5}, \frac{2k(x+w)+4k(y+z)}{5}\right) \\ &= \left(\frac{kx+2ky}{5}+\frac{kw+2kz}{5}, \frac{2kx+4ky}{5}+\frac{2kw+4kz}{5}\right) \\ &= \left(\frac{kx+2ky}{5}, \frac{2kx+4ky}{5}\right)+\left(\frac{kw+2kz}{5}, \frac{2kw+4kz}{5}\right) \\ &= k(x,y)+k(w,z), \end{aligned}$$

so this axiom holds.

8) We have

$$(k+m)(x,y)=\left(\frac{(k+m)x+2(k+m)y}{5},\frac{2(k+m)x+4(k+m)y}{5}\right)$$
$$=\left(\frac{kx+2ky}{5}+\frac{mx+2my}{5},\frac{2kx+4ky}{5}+\frac{2mx+4my}{5}\right)$$
$$=\left(\frac{kx+2ky}{5},\frac{2kx+4ky}{5}\right)+\left(\frac{mx+2my}{5},\frac{2mx+4my}{5}\right)$$
$$=k(x,y)+m(x,y),$$

so this axiom holds

9) We have

$$k(m(x,y))=k\left(\frac{mx+2my}{5},\frac{2mx+4my}{5}\right)$$
$$=\left(\frac{k\left(\frac{mx+2my}{5}\right)+2k\left(\frac{2mx+4my}{5}\right)}{5},\frac{2k\left(\frac{mx+2my}{5}\right)+4k\left(\frac{2mx+4my}{5}\right)}{5}\right) \quad (6)$$
$$=\left(\frac{\frac{5kmx}{5}+\frac{2\cdot 5kmy}{5}}{5},\frac{\frac{2\cdot 5kmx}{5}+\frac{4\cdot 5kmy}{5}}{5}\right)$$
$$=\left(\frac{(km)x+2(km)y}{5},\frac{2(km)x+4(km)y}{5}\right)=(km)(x,y),$$

so this axiom holds.

10) We have

$$1(x,y):=\left(\frac{1\cdot x+2\cdot 1\cdot y}{5},\frac{2\cdot 1\cdot x+4\cdot 1\cdot y}{5}\right)=\left(\frac{x+2y}{5},\frac{2x+4y}{5}\right)$$

which is not (x,y) in every instance. For example, $1(2,-1)=\left(\frac{2+2\cdot(-1)}{5},\frac{2\cdot 2+4\cdot(-1)}{5}\right)=(0,0)\neq(2,-1)$. Thus all axioms hold except the last.

5) Prove that for any (real) vector space $\left(V, \mathbb{R}, +, \bullet\right)$ (satisfying the ten axioms)—no matter how bizarre the addition $+$ and the scalar multiplication $\bullet$ are—we must have $0\bullet\mathbf{u} = \mathbf{z}$ for any vector $\mathbf{u} \in V$ ($\in \mathbb{R}$), where $\mathbf{z}$ is the "zero" vector in the space, i.e. where $\mathbf{z}$ is the additive identity in V. Be sure to list the axioms used in your proof. Feel free to use the fact that

$$\begin{aligned} &\mathbf{w} + \mathbf{v} = \mathbf{v} \Rightarrow \mathbf{w} = \mathbf{z}, \text{ or} \\ &\mathbf{v} + \mathbf{w} = \mathbf{v} \Rightarrow \mathbf{w} = \mathbf{z}, \end{aligned} \tag{7}$$

i.e. that if a vector $\mathbf{w}$ acts like $\mathbf{z}$ even for just one $\mathbf{v} \in V$, then it is $\mathbf{z}$. On the other hand, you may also do what you did in the relevant type of homework problems (which invents the fact indicated in equation (7) for you).

15pts

Solution

By axiom 8)

$$0\bullet\mathbf{u} + 0\bullet\mathbf{u} = \left(0+0\right)\bullet\mathbf{u}, \tag{8}$$

which, by property of the number $0 \in \mathbb{R}$, gives

$$0\bullet\mathbf{u} + 0\bullet\mathbf{u} = 0\bullet\mathbf{u}. \tag{9}$$

But now this is the left-hand side of equation (7) above with $\mathbf{w} = 0\bullet\mathbf{u}$ (and, less important, $\mathbf{v} = 0\bullet\mathbf{u}$). So by the right-hand side of equation (7) $\mathbf{w} = 0\bullet\mathbf{u} = \mathbf{z}$.

6) By use of the relevant "if and only if" theorem, determine whether the following is a subspace of M_{nn}: (M_{nn} is the vector space of $n \times n$ matrices with ordinary matrix addition and scalar multiplication.) the set W of all $n \times n$ matrices A such that $A^T = -A$. MAKE SURE AND REFERENCE AND **USE** THE THEOREM in determining your conclusion. Either way, prove your conclusion. Note that we have chosen here

$$W = \left\{ A \in M_{nn} \middle| A^T = -A \right\}. \tag{10}$$

In referencing the theorem, it might be helpful to refer to W.

20pts

Solution

The theorem is as follows: Let W be a non-empty ***subset*** of elements of a real vector space V . Then W is, in addition, a (real) ***subspace*** of V iff

$$c,k \in \mathbb{R},\ \mathbf{u},\mathbf{v} \in W \Rightarrow c\mathbf{u} + k\mathbf{v} \in W. \tag{11}$$

Changing the notation in (11) to be more traditional for matrices (as in (10)) we could write (11) as

$$c,k \in \mathbb{R},\ A, A' \in W \Rightarrow cA + kA' \in W. \tag{12}$$

To check that our hypotheses hence conclusion of this theorem hold, we first note that the subset W defined by (10) is nonempty: if nothing else $A = 0$ (the zero matrix) is “skew”, i.e. satisfies $A^T = -A$, so that $W \supset \{0 \in M_{nn}\} \neq \varnothing$, where $\varnothing$ is notation for the empty set. (A square zero matrix is also symmetric, i.e. $A^T = A$ for $A = 0 \in M_{nn}$, but this is not relevant.) To show (12) always holds and, so, to show the set W is actually a subspace of M_{nn} we note that when A and A' are both in subset W (giving both $A^T = -A$ and $A'^T = -A'$), and when c,k are arbitrary real numbers, we have $cA + kA'$ is also in W because

$$\left(cA + kA'\right)^T = cA^T + kA'^T = c\left(-A\right) + k\left(-A'\right) = -\left(cA + kA'\right). \tag{13}$$

In (13) we used, in order, properties of the transpose, membership of A and A' in subset W and finally properties of matrix algebra.

7) Determine whether the following statement is (always) true or (sometimes) false: “If $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$ is a linearly dependent (nonempty) set of vectors from a vector space V , then so is the set $S' = \{\mathbf{v}_1, \ldots, \mathbf{v}_r, \mathbf{v}_{r+1}\}$, provided $\mathbf{v}_{r+1} \in V$.” If it is true, prove it, otherwise give a counter example.

<u>20pts</u>

<u>Solution</u>

The statement is (always) true: Since $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$ is linearly dependent, by definition there exists real r -tuple $\left(k_1, \ldots, k_r\right) \neq \left(0, \ldots, 0\right) \in \mathbb{R}^r$ such that

$$k_1\mathbf{v}_1 + \ldots + k_r\mathbf{v}_r = \mathbf{z}, \tag{14}$$

where $\mathbf{z}$ denotes the “zero” vector in the relevant vector space. So then the equation

$$c_1\mathbf{v}_1 + \ldots + c_r\mathbf{v}_r + c_{r+1}\mathbf{v}_{r+1} = \mathbf{z} \tag{15}$$

also holds for the non-trivial $r+1$-tuple $(c_1,\ldots,c_r,c_{r+1}) = (k_1,\ldots,k_r,0) \neq (0,\ldots,0,0) \in \mathbb{R}^{r+1}$: with this choice we have

$$c_1\mathbf{v}_1 + \ldots + c_r\mathbf{v}_r + c_{r+1}\mathbf{v}_{r+1} = k_1\mathbf{v}_1 + \ldots + k_r\mathbf{v}_r + 0\mathbf{v}_{r+1} = k_1\mathbf{v}_1 + \ldots + k_r\mathbf{v}_r + \mathbf{z} = k_1\mathbf{v}_1 + \ldots + k_r\mathbf{v}_r = \mathbf{z}. \tag{16}$$

Here we used the result of problem 5 (in the form $0\mathbf{v}_{r+1} = \mathbf{z}$), together with axiom 4 (regarding the action of the zero vector on other vectors by addition), and then finally hypothesis (14). Thus, definition, S' is linearly dependent as claimed (and the statement is always true).

8) Find the coordinate vector of $\mathbf{w}$ relative to basis $S = \{\mathbf{u}_1, \mathbf{u}_2\} \subset P_1$:

$$\mathbf{u}_1 = 1 + x, \qquad \mathbf{u}_2 = 2x, \qquad \mathbf{w} = a + bx. \tag{17}$$

(P_1 is the vector space of linear functions, with vector addition and scalar multiplication being the ordinary operations on functions.)

15pts

Solution

$$(\mathbf{w})_S = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \tag{18}$$

is the coordinate vector of $\mathbf{w}$ relative to basis $S = \{\mathbf{u}_1, \mathbf{u}_2\}$ iff

$$\alpha\mathbf{u}_1 + \beta\mathbf{u}_2 = \mathbf{w} \ , \tag{19}$$

i.e. iff

$$\alpha(1+x) + \beta(2x) = a + bx, \tag{20}$$

which, by the rules of algebra on the vector space P_1, can be rewritten to emphasize P_1's standard basis $\{1, x\}$ and its independence as follows:

$$(\alpha - a)\cdot 1 + (\alpha + 2\beta - b)\cdot x = 0, \tag{21}$$

where the right hand side of (21) is to be thought of as the zero function, i.e. the zero vector in P_1. Since $\{1, x\}$ is linearly independent, by definition (21) holds iff

$$\alpha - a = \alpha + 2\beta - b = 0, \tag{22}$$

the zero here being the ordinary one in the reals. Easily we get (22) holds iff $\alpha = a,\ \beta = (b-a)/2$, i.e. iff

$$(\mathbf{w})_S = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} a \\ (b-a)/2 \end{pmatrix}. \tag{23}$$

Indeed we confirm that with (23) we have

$$\begin{aligned} \alpha(1+x) + \beta(2x) &= a(1+x) + \frac{b-a}{2}(2x) = a(1+x) + (b-a)x = a \cdot 1 + bx \\ &= a + bx, \end{aligned} \tag{24}$$

as required by (20).

9) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be linear, and suppose

$$T\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix},\ T\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}. \tag{25}$$

What is $T\begin{pmatrix} 1 \\ 0 \end{pmatrix}$? What is $T\begin{pmatrix} 0 \\ 1 \end{pmatrix}$? What is the standard matrix $[T]$ of T? Make sure you effectively prove this result rather than just “eye balling it”.

15pts

Solution

Since (by inspection—you can work harder if need be)

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = -\frac{1}{3}\begin{pmatrix} 1 \\ 2 \end{pmatrix} + \frac{2}{3}\begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{2}{3}\begin{pmatrix} 1 \\ 2 \end{pmatrix} - \frac{1}{3}\begin{pmatrix} 2 \\ 1 \end{pmatrix}, \tag{26}$$

then, by linearity, and with assumptions (25), we have

$$T\begin{pmatrix}1\\0\end{pmatrix}=T\left(-\frac{1}{3}\begin{pmatrix}1\\2\end{pmatrix}+\frac{2}{3}\begin{pmatrix}2\\1\end{pmatrix}\right)=-\frac{1}{3}T\begin{pmatrix}1\\2\end{pmatrix}+\frac{2}{3}T\begin{pmatrix}2\\1\end{pmatrix}=-\frac{1}{3}\begin{pmatrix}2\\1\end{pmatrix}+\frac{2}{3}\begin{pmatrix}1\\2\end{pmatrix}=\begin{pmatrix}0\\1\end{pmatrix}, \quad \text{and}$$
$$T\begin{pmatrix}0\\1\end{pmatrix}=T\left(\frac{2}{3}\begin{pmatrix}1\\2\end{pmatrix}-\frac{1}{3}\begin{pmatrix}2\\1\end{pmatrix}\right)=\frac{2}{3}T\begin{pmatrix}1\\2\end{pmatrix}-\frac{1}{3}T\begin{pmatrix}2\\1\end{pmatrix}=\frac{2}{3}\begin{pmatrix}2\\1\end{pmatrix}-\frac{1}{3}\begin{pmatrix}1\\2\end{pmatrix}=\begin{pmatrix}1\\0\end{pmatrix}. \tag{27}$$

One may have intuited this, but this is a rigorous way to show the result. By theorem then the standard matrix $[T]$ is given by

$$[T]=\left[\; T\begin{pmatrix}1\\0\end{pmatrix} \middle| \; T\begin{pmatrix}0\\1\end{pmatrix}\right]=\left[\; \begin{pmatrix}0\\1\end{pmatrix} \middle| \; \begin{pmatrix}1\\0\end{pmatrix}\right]=\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}. \tag{28}$$